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A STUDY OF THE GENERALIZED DENSITY MATRIX IN THE SU(3) MODEL OF ELLIOTT FOR AN ARBITRARY OSCILLATOR *n*-SHELL

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Abstract: The SU(3) model of Elliott for an arbitrary oscillator *n*-shell is considered. Exact solutions corresponding to the low-lying collective SU(3) multiplets are obtained. These multiplets exhibiting distinct collective properties are used as a "collective band". All the matrix elements from the one-particle density matrix operator inside a given band are investigated. The so-formed matrix R, i.e. the generalized density matrix (GDM), is diagonalized and an explicit expression for the eigenvalues of the GDM in the case of low-lying multiplets is found. The GDM diagonal form contains a number of vanishing eigenvalues (the number of such "zero" eigenvalues is equal to that of the occupied orbits in the oscillator *n*-shell). Most of the remaining eigenvalues are close to unity. The asymptotic behaviour of the nonzero eigenvalues is analyzed in the limit of large nucleon number and the accuracy of the normalization condition $R^2 = R$ is estimated.

1. Introduction

Recently essential progress in the phenomenological description of the collective properties of atomic nuclei has been achieved in the framework of the interacting boson model [1]). In this model the low-lying collective states can be expressed through a finite number of boson creation operators. For this purpose group theory methods can be used very effectively. However, the agreement between theory and experiment is reached by using a large number of fitting parameters, being individual for each nucleus. In this case the importance of microscopic approaches to the collective excitation description has increased since they can form a microscopic basis of the interacting boson model and predict its phenomenological parameters. From our viewpoint the method of the generalized density matrix (GDM) [2]) is very promising. Earlier the logical consistency and effectiveness of this method were demonstrated [3]) both for the spherical and deformed nuclei. However, for grounding the interacting boson model, other approximations are necessary. In this connection the equations of the GDM method must be studied in detail and their completeness and consistency must be analyzed. For the investigation of such basic problems and elaboration of approximate approaches exactly soluble models are fruitful. Earlier in ref. [4]) Janssen

obtained the GDM R in the Elliott model [5]) for a space of single-particle states with small dimension (the ^{20}Ne nucleus). This restriction was removed in ref. [6]) for the Lipkin model. The R-matrix was obtained for the arbitrary dimension of the single-particle space in the frame of the above model. The results of this paper allow us to assume that the violation of the $R^2 = R$ condition found by Janssen was due to the small volume of the single-particle space used. However, the Lipkin model is still rather specific and restricted. Therefore, it would be desirable to have an exact solution of the GDM for a more sophisticated and realistic model. In our opinion the Elliott model with a large number of nucleons occupying an arbitrary harmonic oscillator shell is convenient for this purpose.

In this paper the following points are dealt with:

(i) the SU(3) model of Elliott for an arbitrary nuclear shell is considered;

(ii) exact solutions corresponding to the low-lying collective SU(3) multiplets are obtained;

(iii) explicit expressions for the eigenvalues of the GDM in the case of the above-mentioned SU(3) multiplets are obtained;

(iv) the asymptotical behaviour of the eigenvalues is analyzed in the limit of large A;

(v) the accuracy of the normalization condition $R^2 = R$ is estimated.

2. The density matrix and its determination

2.1. THE INTERACTING BOSON MODEL AND THE GDM

The microscopic foundation of the interacting boson method assumes the establishment of a connection between the boson space $|\Lambda\rangle$ with s- and d-boson operators and that of the collective states $|\Lambda)$ and collective operators of the fermion system. Recently intensive investigations concerning the construction of the collective operators (as a rule, in terms of bifermion operators) were carried out, in particular, by theoretical groups in Japan [7]). However, it is possible to consider the set of matrix elements

$$\langle \Lambda\lambda | R | \Lambda'\lambda' \rangle \equiv (\Lambda | a_\lambda a^+_{\lambda'} | \Lambda') \tag{2.1}$$

defining the generalized density matrix (GDM) R instead of constructing the collective operators *a priori*.

Eq. (2.1) makes clear the correspondence between the collective "band" $|\Lambda)$ of the fermion system and the model collective space $|\Lambda\rangle$[†]. Thus, R is a matrix both in a single-particle space $|\lambda\rangle$ and collective one $|\Lambda\rangle$, simultaneously.

[†] The collective "band" means a set of levels connected to each other by strong multipole transitions (direct or cascade) and displaying hindered transitions into states of different physical nature.

The arbitrary bifermion operator $\hat{Q} = \sum_{\lambda\lambda'} q_{\lambda\lambda'} a^+_\lambda a_{\lambda'}$ connecting the states belonging to the same collective band has the following form in the model space:

$$Q = -\mathrm{Tr}\,(qR)\,, \tag{2.2}$$

where the symbol Tr implies the summation over the single-particle space only.

A conventional density matrix can be determined as an average value of $a_\lambda a^+_{\lambda'}$,

$$\rho_{\lambda\lambda'} = (0|a_\lambda a^+_{\lambda'}|0)\,,$$

at a definite state of the system (the ground state $|0)$, as a rule). In the Hartree–Fock approximation the equation for ρ has the form

$$[h(\rho), \rho] = 0\,, \tag{2.3}$$

where $h(\rho)$ is a single-particle hamiltonian including the self-consistent field potential. It is obtained by the closure of the equations of motion for $a_\lambda a^+_{\lambda'}$ inside the single state $|0)$, i.e. by neglecting the nondiagonal elements $(0|a_\lambda a^+_{\lambda'}|c)$.

It is natural to derive the equation for R by closing the set of equations of motion for $a_\lambda a^+_{\lambda'}$ inside a single "band" $|\Lambda)$, neglecting the transitions into states of different physical nature. As a result, we obtain the equation [2])

$$[H + h(R), R] = 0\,, \tag{2.4}$$

which is similar to eq. (2.3) but in the combined space $|\lambda\rangle\otimes|\Lambda)$. Here H is a collective hamiltonian (in the $|\Lambda)$ space). Its eigenvalues are equal to the energies of collective states in the considered "band" $|\Lambda)$; $h(R)$ is a self-consistent single-particle hamiltonian.

The general structure of eq. (2.4) can be understood in the frame of perturbation theory with respect to H. In zero approximation this equation coincides with eq. (2.3), whose solution either does not depend on collective variables at all or depends only on the "collective" coordinates (but not on the collective momenta). In this case the collective coordinates will only be parameters of the solution $R^{(0)}$. In the first approximation we obtain either the RPA equations (in the first case) or the cranking model ones (in the second case). The first-order expression for the R-matrix can be found unambiguously if the zero-order R-matrix is determined by eq. (2.3).

2.2. AN UNAMBIGUOUS DETERMINATION OF THE CONVENTIONAL DENSITY MATRIX

Let us consider the problem of unambiguous determination of the matrix ρ from eq. (2.3). Formally, this equation can be satisfied by the arbitrary matrix

$$\langle\lambda|\rho|\lambda'\rangle = \sum_\nu \langle\lambda|\nu\rangle\rho_\nu\langle\nu|\lambda'\rangle\,, \tag{2.5}$$

where the state $|\nu\rangle$ is an eigenvector of the hamiltonian h and the eigenvalues ρ_ν are arbitrary. But, really, the allowed values ρ_ν are restricted by some special

conditions. Namely, the limitations on the average values of occupation numbers must be fulfilled,

$$0 \leqslant \langle \lambda | \rho | \lambda \rangle \leqslant 1 , \tag{2.6}$$

which are the consequences of the Pauli principle. Then, the hamiltonian $h(\rho)$ must be consistent with ρ. However, these conditions still give us a rather wide freedom in choosing ρ_ν. Therefore, additional physical considerations are required. In particular, in the Hartree–Fock approximation, the many-particle wave function has the form of a single Slater determinant, i.e. the occupation numbers (2.6) of single-particle states are equal to 0 or 1, which is equivalent to the normalization condition $\rho^2 = \rho$. However, the distribution of "zeros" and "units" among single-particle states has not been determined yet. Therefore, the structure of the ground state should be fixed (for example, all single-particle orbits inside the Fermi sphere are occupied in the ground state of the system, etc.).

2.3. THE PROBLEM OF UNAMBIGUOUS DETERMINATION OF THE GDM

Formally, eq. (2.4) is similar to eq. (2.3) and its solution can be represented in the form of eq. (2.5):

$$\langle \lambda \Lambda | R | \lambda' \Lambda' \rangle = \sum_{\mathscr{L}} \langle \lambda \Lambda | \mathscr{L} \rangle \boldsymbol{r}_{\mathscr{L}} \langle \mathscr{L} | \lambda' \Lambda' \rangle , \tag{2.7}$$

where $|\mathscr{L}\rangle$ stands for the eigenvectors of the hamiltonian $H + h$ in the combined space $|\lambda\rangle \otimes |\Lambda\rangle$, while the parameters $\boldsymbol{r}_{\mathscr{L}}$ are arbitrary.

Although the eigenvalues $\boldsymbol{r}_{\mathscr{L}}$ are formally similar to the occupation numbers ρ_ν in eq. (2.5), their meaning is not so clear in a general case. Therefore, their unambiguous determination is impossible without using some additional physical reasoning.

First let us consider some sum rules. The diagonal element $\langle \lambda \Lambda | R | \lambda \Lambda \rangle$ is equivalent to the occupation number of the single-particle state $|\lambda\rangle$ when the system is in the state $|\Lambda\rangle$. Therefore, the following condition can be written

$$0 \leqslant \sum_{\mathscr{L}} |\langle \lambda \Lambda | \mathscr{L} \rangle|^2 \boldsymbol{r}_{\mathscr{L}} \leqslant 1 \tag{2.8}$$

on the basis of eq. (2.7).

For the total number of particles (in the fixed state $|\Lambda\rangle$) we have

$$\Omega - A = \sum_{\lambda, \mathscr{L}} |\langle \lambda \Lambda | \mathscr{L} \rangle|^2 \boldsymbol{r}_{\mathscr{L}} . \tag{2.9}$$

After the summation of eq. (2.9) over the states $|\Lambda\rangle$ and using the unitary property of the coefficients $\langle \lambda \Lambda | \mathscr{L} \rangle$ we obtain

$$(\Omega - A)\omega = \sum_{\mathscr{L}} \boldsymbol{r}_{\mathscr{L}} . \tag{2.10}$$

In eqs. (2.9), (2.10) the quantities Ω and ω are the dimensions of the single-particle and collective spaces, respectively. They are assumed to be finite for simplicity.

Another type of limitation for the R-matrix is connected with the consistency properties of the collective operators. Let us calculate the commutator between the operator (2.3) and $a_\lambda a_\lambda^+$ and obtain the matrix element of this equation between the states of the collective band. If the operator $\hat{Q}$ is collective, its matrix elements are highly enhanced inside the band under consideration (in fact, it is the collective operator definition) and the following relation takes place:

$$[Q+q, R]=0\,, \tag{2.11}$$

where the operator Q is given by eq. (2.2). The total number of equations (2.11) is equal to that of collective operators that can be introduced in the collective space $|\Lambda\rangle$.

Thus, in addition to the dynamic equation (2.4), the sum rules (2.8)-(2.10) and the consistency conditions (2.11) can be applied to the R-matrix determination, too. Obviously, the normalization condition $R^2=R$ (if it is valid) puts additional limitations on the R-matrix.

3. The SU(3) model of Elliott for an arbitrary nuclear shell

In the nuclear shell model suggested by Elliott [5]) the A nucleons occupy the degenerate single-particle levels with quantum number n in the spherical harmonic oscillator potential. The hamiltonian of this system includes separable quadrupole-quadrupole interaction between the nucleons and has the form

$$\mathcal{H}=-\tfrac{1}{2}\kappa\sum_{m=-2}^{2}(-1)^m\hat{Q}_m\hat{Q}_{-m}\,. \tag{3.1}$$

The quadrupole momentum operators $\hat{Q}$ can be expressed as

$$\hat{Q}_m=\sum_{1,2}(q_m)_{12}a_1^+a_2\,, \tag{3.2}$$

where $|1\rangle$ is the state with quantum numbers n_x, n_y, n_z, σ, τ of the spherical harmonic oscillator potential (n_x, n_y, n_z ($n_x+n_y+n_z=n$) are the numbers of oscillator quanta along the x, y, z axes, respectively; σ, τ are the nucleon spin and isospin projections).

The number of different single-particle orbits in the n-shell is equal to $\Omega=\frac{1}{2}(n+1)(n+2)$. Hereafter we shall assume that these orbits are occupied by two sorts of nucleons: protons and neutrons. The total number of single-particle states for the proton-neutron system is equal to 4Ω (it will be referred to as a single-particle space dimension). The five operators (3.2) and the operators of the orbital angular momentum for the A-nucleon system L_i ($i=x, y, z$) can be treated as the SU(3) group generators. If we add the operator of the total number of oscillator quanta $N=n\sum_1 a_1^+a_1$, the nine operators $\hat{Q}_m$, L_i, N form the Lie algebra of the U(3) group.

The totally antisymmetric states of the A nucleons can be labelled by the Young scheme $[f]=[f_1, f_2, \ldots, f_m]$ where $m<\Omega, f_1 \geq f_2 \geq \cdots \geq f_m \geq 0, f_1+f_2+\cdots+f_m=A$, and all f_i are nonnegative integers. This Young scheme presents the irreducible representation (IR) of the U(Ω) group (the unitary transformations of the single-particle orbits) and characterizes the state symmetry with respect to the permutation group S_A of the space variables of the A particles, simultaneously. The IR of the spin-isospin $U_{ST}(4)$ group is given by the dual Young scheme $[\tilde{f}]$ which can be obtained from $[f]$ by exchanging the rows with columns. For the even-even nuclei with $Z=N$ and $A=4k$ the Young schemes have the form $[f]=[4^k]$, $[\tilde{f}]=[k^4]$ for the lowest states (k is the number of occupied single-particle orbits). The spin-isospin content of the Wigner supermultiplet can be obtained by standard algebraic methods [see, e.g. refs. [8,9])]. For our purpose it is enough to remark that only one combination, $S=T=0$, is consistent with $[f]=[4^k]$, and only $S=T=\frac{1}{2}$ is allowed in the Wigner supermultiplet $[f']=[4^k 1]$.

It is reasonable to classify the coordinate part of the system wave function in correspondence with the chain of subgroups U(Ω) $\supset$ U(3) $\supset$ SU(3) $\supset$ SO(3) [ref. [5])]. The IR of the U(3) group is characterized by the Young scheme $\{N_H\}=\{N_{zH}, N_{xH}, N_{yH}\}$, where $N_{zH} \geq N_{xH} \geq N_{zH} \geq 0$ are integers, $N_{zH}+N_{xH}+N_{yH}=nA$. The IR of the SU(3) group are usually enumerated by the Elliott symbol $(\lambda\mu)$ where

$$\lambda = N_{zH} - N_{xH}, \qquad \mu = N_{xH} - N_{yH}. \tag{3.3}$$

A set of states transformed in correspondence with the IR $(\lambda\mu)$ of the SU(3) group will be referred to as an SU(3) multiplet. Each multiplet $(\lambda\mu)$ contains $\dim(\lambda\mu)=\frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$ states. This set of states can be represented as a system of rotational bands characterized by some quantum number K taking the values

$$K=\mu, \mu-2, \ldots, 1 \quad \text{or} \quad 0. \tag{3.4}$$

The orbital angular momentum L takes the values $L=K, K+1, \ldots, K+\lambda$ for $K\neq 0$ and $L=\lambda, \lambda-2, \ldots, 1$ or 0 for $K=0$.

The index K indicates the way of constructing a state of A nucleons in the oscillator shell n with a given momentum L and projection M,

$$|(n0)^A, [4^k], (\lambda\mu)LMK, SM_S, TM_T\rangle = \hat{P}^L_{MK}\chi_H, \tag{3.5}$$

where $\hat{P}^L_{MK}$ is the projection operator on a state with given L and M. Below the fixed quantum numbers $(n0)^A$, $[f]$, S, M_S and T, M_T in the wave function (3.5) are omitted for simplicity. The vector χ_H determining the structure of the state (3.5) is the highest-weight vector of the IR $\{N_H\}$ of the U(3) group. It means that the following conditions are satisfied:

$$A_{zx}\chi_H = A_{zy}\chi_H = A_{xy}\chi_H = 0, \tag{3.6}$$

where the operators

$$A_{ik} = \sum_{1,2} \langle 1|b_i^+ b_k|2\rangle a_1^+ a_2, \tag{3.7}$$

are the operators of the U(3) group taken in a standard Cartan-Weyl form. The following commutation rules,

$$[A_{ik}, A_{pq}] = \delta_{kp}A_{iq} - \delta_{iq}A_{pk}\,, \tag{3.8}$$

are valid for them. In eq. (3.7) b_i^+ and b_i are the usual creation and annihilation operators of the oscillator quanta along the i-axes.

The vector χ_H is characterized by a definite number of oscillator quanta N_{zH}, N_{xH}, N_{yH} along the z, x, y axes: $A_{ii}\chi_H = N_{iH}\chi_H$. The states (3.5) form the basis of the IR $\{N_H\}$ of the U(3) group. They are nonorthogonal with respect to the quantum number K. However, the non-orthogonality decreases quickly with increasing λ and μ (one or both the values) [see refs. [5,10,11])]. Therefore, it is possible to treat K as a good quantum number asymptotically. The basis (3.5) corresponding to the reduction SU(3) $\supset$ SO(3) is called the Elliott basis, unlike the canonical Gelfand-Tseitlin one with quantum numbers N_x, N_y, N_z. The technique of calculations in this basis is described in refs. [5,10]). In particular, the energy of a state with given L belonging to the SU(3) multiplet $(\lambda\mu)$ can be written in the form

$$E_{(\lambda\mu)L} = -3\kappa g_2(\lambda\mu) + \tfrac{3}{2}\kappa L(L+1)\,, \tag{3.9}$$

where

$$g_2(\lambda\mu) = \tfrac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \tag{3.10}$$

is an eigenvalue of the second-order Casimir operator for the SU(3) group. Note that the states with the same L belonging to different K are degenerated.

In the Elliott shell model the electric quadrupole transitions are enhanced to the maximal degree possible in taking into account the configuration mixture in an oscillator shell. In fact, the operator of the electric quadrupole momentum is equal to $\frac{1}{2}e\hat{Q}_m$, being the generator of the SU(3) group. Therefore, the nonvanishing E2 transitions take place only between states belonging to one and the same SU(3) multiplet. The quadrupole sum rule $\sum_f |\langle f\|\hat{Q}\|0\rangle|^2$ for transitions from the ground state $|0\rangle$ of the multiplet $(\lambda\mu)$ is proportional to the g_2 (3.10). The ratio of reduced probabilities of the E2 transitions with $\Delta K = \pm 2$ and $\Delta K = 0$ can be estimated as follows:

$$\frac{B(\mathrm{E2}, \Delta K = \pm 2)}{B(\mathrm{E2}, \Delta K = 0)} \sim \mu/(2\lambda + \mu)\,. \tag{3.11}$$

For the internal quadrupole momentum components

$$Q_0 = \mathring{Q}\cos\gamma \sim 2\lambda + \mu\,, \qquad Q_{\pm 2} = \sqrt{\tfrac{1}{2}}\,\mathring{Q}\sin\gamma \sim \mu\,. \tag{3.12}$$

The expressions (3.11) and (3.12) show that the space of SU(3) multiplet states is similar to the system of nonaxial rotational bands at $\lambda \sim \mu \gg 1$. In the case of $\lambda \gg \mu$ this space splits into a number of rotational bands with different values of K which are connected by hindered interband transitions (in comparison with strong intraband E2 transitions).

3.1. NORMAL OCCUPATION OF SINGLE-PARTICLE OSCILLATOR ORBITS

The highest-weight vector χ_H can be constructed by the occupation of the oscillator orbits $(n_x n_y n_z)$ by the A nucleons. Let us introduce the following ordering of the orbits in the nth harmonic oscillator shell:

$$
\begin{array}{ccccccccc}
(00n) & \to & (10n-1) & \to & (20n-2) & \to & (30n-3) & \to \cdots \to & (n00) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(01n-1) & \to & (11n-2) & \to & (21n-3) & \to & \cdots & \to & (n-110) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (02n-2) & \to & (12n-3) & \to & \cdots & \to & (n-220) \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (03n-3) & \to & \cdots & \to & (n-330) \\
 & & & & & & \vdots & & \downarrow \\
 & & & & & & \vdots & \to & \vdots \\
 & & & & & & \vdots & & \downarrow \\
 & & & & & & & & (0n0) .
\end{array}
\tag{3.13}
$$

The total set of the A-nucleon system states (its number is equal to $\binom{4\Omega}{A}$) can be constructed by distributing the A nucleons among the orbits. In order to obtain a state with $[f]=[4^k]$ and $S=T=0$, four nucleons should be placed on each of the chosen orbits $(n_{xi}n_{yi}n_{zi})$, $i=1,\ldots,k$ with different values of the spin and isospin projections σ and τ. The corresponding wave function has the form of a Slater determinant:

$$
\chi = \prod_{i=1}^{k} \prod_{\sigma,\tau} a^{+}_{(n_{xi}n_{yi}n_{zi}),\sigma,\tau}|0\rangle . \tag{3.14}
$$

In general, the highest-weight vector of some IR of the SU(3) group is a linear combination of determinants (3.14) with the same values of $N_z = N_{z\mathrm{H}}$, $N_x = N_{x\mathrm{H}}$, $N_y = N_{y\mathrm{H}}$. However, the Slater determinant corresponding to the so-called *normal* occupation of orbits can be taken as the highest-weight vector. The normal occupation stands for such a distribution of the A nucleons among the orbits (3.13) when, together with any occupied orbit $(n_x n_y n_z)$, the preceding ones $(n_x - 1 n_y n_z)$ and $(n_x n_y - 1 n_z)$ are filled, too. In the triangle (3.13) each preceding orbit is connected with the succeeding ones by arrows.

The wave function corresponding to the normal occupation is the highest-weight vector of the IR $\{N_H\}$. In fact, operating on this function by the raising generators A_{zx}, A_{zy} and A_{xy} one can easily prove the equalities (3.6). It is obvious that

$$
N_{\alpha\mathrm{H}} = 4 \sum_{i=1}^{k} n_{\alpha i} , \qquad \alpha = x, y, z . \tag{3.15}
$$

The advantage of normal occupation does not consist only in a simple structure of the highest-weight vector. It can be proved that the normal occupation corresponds

to the SU(3) multiplets with largest eigenvalues (3.10) of the Casimir operator. These multiplets are energetically favourable and display the pronounced collective properties. Further we shall restrict ourselves to a consideration of such multiplets only. This multiplet wave function in the projected Hartree-Fock approximation coincides exactly with the function (3.5), the vector χ_H being the nuclear intrinsic-state wave function. Thus, the nucleon states on the orbits $(n_x n_y n_z)$ can be compared to the asymptotic limit of the Nilsson model with non-axial symmetry.

In this way the eigenfunctions of the energetically lowest SU(3) multiplets of the Elliott model for the arbitrary oscillator n-shell can be determined with the normal occupation of the oscillator orbits. These eigenfunctions are used in appendix A for calculations necessary to obtain eigenvalues of the GDM employing the matrix element of the operator a^+.

3.2. THE LOWEST MULTIPLETS IN THE SU(3) ELLIOTT MODEL

Among various SU(3) multiplets corresponding to the normal occupation there is one which includes the ground state of the nucleus. Let us call this multiplet a *ground* one.

In order to determine the ground multiplet the normal occupation must be found which gives the maximal value of $g_2(\lambda\mu)$ (3.10). With this aim let us first consider the successive occupation of columns in eq. (3.13) from left to right. (Note that we can begin filling the next column from top to bottom only after the total occupation of the previous one.)

Let us assume that in the A-nucleon system ($A < 2\Omega$) all the columns including the $(p+1)$th one are completely filled by nucleons and in the $(p+2)$th column the orbits with $n_y = 0, 1, \ldots, q$ are occupied, too. It is easy to obtain that

$$A = 2(p+1)(p+2) + 4(q+1) \tag{3.16}$$

and the quantum numbers of the SU(3) multiplet are equal to

$$\lambda = 2(p+1)(p+2)(n-p) + 2(q+1)(2n-4p+q-4)\,,$$

$$\mu = 4(q+1)(p-q+1)\,. \tag{3.17}$$

In the case of a large number of particles ($p \gg 1$) the SU(3) multiplet obtained by this type of occupation is close to the IR $(\lambda 0)$. It can be proved that the successive occupation of columns gives the maximum values of $g_2(\lambda\mu)$ for nuclei with $A < 2\Omega$. Therefore, the SU(3) multiplet (3.17) is a ground one for nuclei belonging to the first half of the shell.

In order to determine the ground multiplets for nuclei belonging to the second half of the shell ($A > 2\Omega$), it is necessary to take into account the fact that the allowed SU(3) multiplets for the A-hole nucleus may be obtained by the exchange of λ and μ [ref. [5])]. Since the eigenvalue $g_2(\lambda\mu)$ is invariant with respect to this

exchange, the ground multiplets for the nuclei belonging to the second half of the shell are close to the IR (0μ). Such multiplets can be obtained by the successive occupation of rows in the triangle (3.13).

Thus, we have selected two most energetically favourable types of the normal occupation: the occupation of columns at $A<2\Omega$ and that of rows at $A>2\Omega$. In the limit $A\gg 1$ the energies of lowest levels 0^+ in these multiplets are given by the expressions

$$E_{(\lambda\mu)0^+}(\eta)=-\kappa(4\Omega)^3\eta^2(1-\sqrt{\eta})^2\,,$$
$$\tilde{E}_{(\lambda\mu)0^+}(\eta)=E_{(\lambda\mu)0^+}(1-\eta)\,, \tag{3.18}$$

where the parameter $\eta=A/4\Omega$ $(0\leq\eta\leq 1)$ determines what fraction of the shell is occupied by nucleons. Note that the second formula (3.18) expresses the hole-particle symmetry of the SU(3) shell model of Elliott. Allowing for the hole-particle symmetry we restrict ourselves to the consideration of nuclei with $A<2\Omega$. The SU(3) multiplet with $\lambda\sim\mu$ corresponding to the successive occupation of diagonals perpendicular to the hypotenuse of the triangle (3.13) (beginning from the upper left angle) is an example of the excited multiplet. If the $2s+2$ $(s<\frac{1}{2}n)$ diagonal lines in this triangle are totally occupied, the corresponding SU(3) multiplet for the system of $A=4(s+1)(s+2)$ nucleons is characterized by the quantum numbers

$$\lambda=\tfrac{4}{3}(s+1)(s+2)(3n-5s-3)\,,$$
$$\mu=\tfrac{2}{3}(s+1)(s+2)(2s+3)\,. \tag{3.19}$$

The energy of its lowest level is equal to

$$E_{(\lambda\mu)0^+}(\eta)=-\kappa(4\Omega)^3\eta^2(1-\tfrac{3}{2}\sqrt{2\eta}+\tfrac{7}{6}\eta)\,. \tag{3.20}$$

For nuclei of the second half of the shell $(s>\frac{1}{2}n)$ the exchange $\eta\to 1-\eta$ must be made in the last formula.

The ground multiplet (3.17) shows the subshell structure since the four-nucleon binding energy suffers "jumps" when the occupation of one column is finished and that of the next column begins. Similar subshell structure is observed in the Hartree-Fock single-particle energies too [12]).

The developed subshell structure is absent from the SU(3) multiplet (3.19) because the binding energy of the four nucleons changes smoothly in transition from filling one diagonal line to another if the distribution of their occupation is changed according to fig. 2.

4. The GDM in the SU(3) model of Elliott

Let us consider an even-even nucleus with $A=4k$ valent nucleons in the nth harmonic oscillator shell. We shall treat all the states of the SU(3) multiplet $(\lambda\mu)$ corresponding to the normal occupation of the oscillator orbits as the "collective

band". In this case the model collective space $|\Lambda\rangle = |(\lambda\mu)LMK\rangle$ coincides with that of the collective states (3.5) of the Fermi system. Making use of the definition (2.1) of the GDM and omitting the fixed quantum numbers n, S and T, we obtain

$$\begin{aligned}\langle (\lambda\mu)L'M'K', l'm'\sigma'\tau'|R|(\lambda\mu)LMK, lm\sigma\tau\rangle \\ = \sum_{\substack{(\lambda''\mu'')\\ L''M''K''}} \langle (\lambda\mu)L'M'K'|a_{l'm'\sigma'\tau'}|(\lambda'\mu')L''M''K''\rangle \\ \times \langle (\lambda'\mu')L''K''M''|a^{+}_{lm\sigma\tau}|(\lambda\mu)LMK\rangle . \end{aligned} \tag{4.1}$$

In this expression the summation is performed over all the intermediate $A+1$ particle states $|(n0)^{A+1}, [4^k 1], (\lambda'\mu')L''M''K'', \frac{1}{2}\sigma, \frac{1}{2}\tau\rangle$.

Let us substitute the matrix elements of the operators a^{+} and a, obtained in appendix A (eq. (A.6)), in (4.1). Then let us multiply the GDM through by the quantities I_3, I_3' (A.3) with fixed values of $(\lambda\mu)$, $(\lambda'\mu')$ and I_4, I_4' (A.4) and sum over LMK, $L'M'K'$, $lm\sigma\tau$, $l'm'\sigma'\tau'$. By allowing for orthonormality properties of the Clebsch-Gordan coefficients the GDM can be reduced to a diagonal form with the eigenvalues

$$r_{(\lambda'\mu')} = \tfrac{1}{16}(A+16)\langle (n0)^{A+1}, [4^k 1], (\lambda'\mu')|(n0)^{A}, [4^k], (\lambda\mu); (n0)\rangle^2 , \tag{4.2}$$

where $\langle \cdots \rangle$ is an orbital single-particle coefficient of fractional parentage.

The GDM eigenvalues (4.2) are degenerated on the quantum numbers $L'M'K'$ since the matrix R, according to the relation (2.11), is a scalar relative to the SU(3) group generators. The multiplicity of each eigenvalue is equal to $4 \dim (\lambda'\mu')$. The explicit form of allowed $(\lambda'\mu')$ symbols can be found by expansion of the direct product of the U(3) IR's into irreducible components:

$$\{N_{z\mathrm{H}}, N_{x\mathrm{H}}, N_{y\mathrm{H}}\} \otimes \{\mathrm{n}00\} = \sum \{N'_{z\mathrm{H}}, N'_{x\mathrm{H}}, N'_{y\mathrm{H}}\} , \tag{4.3}$$

where $N'_{x\mathrm{H}} = N_{x\mathrm{H}} + n'_x \leqslant N_{z\mathrm{H}}$, $N'_{y\mathrm{H}} = N_{y\mathrm{H}} + n'_y \leqslant N_{x\mathrm{H}}$, $N'_{z\mathrm{H}} = N_{z\mathrm{H}} + n'_z$, and $n'_x + n'_y + n'_z = n$.

The quantity (4.2) depends on $(\lambda'\mu')$, $(\lambda\mu)$ only. It is independent of the explicit form of the SU(3) group IR basis. Therefore, in order to calculate the eigenvalues $r_{(\lambda'\mu')}$, it is convenient to use the canonical basis rather than the Elliott one. The matrix element of a^{+} in this basis is obtained in appendix A (eq. (A.7)). Let us use the highest-weight vectors $|(\lambda\mu)\mathrm{H}\rangle$, $|(\lambda'\mu')\mathrm{H}'\rangle$ in this matrix element and exclude the fractional parentage coefficient from eq. (4.2) with the aid of this element. As a result, we obtain the following formula for the practical evaluation of the GDM eigenvalues:

$$\begin{aligned} r_{(\lambda'\mu')} &= I(\lambda'\mu')/J(\lambda'\mu') \\ &= \frac{\langle (n0)^{A+1}, [4^k 1], (\lambda'\mu')\mathrm{H}'|a^{+}_{n'_x n'_y n'_z}|(n0)^{A}, [4^k], (\lambda\mu)\mathrm{H}\rangle^2}{\left\langle \begin{matrix}(\lambda\mu) & (n0) \\ \mathrm{H} & n'_x n'_y n'_z\end{matrix} \middle| \begin{matrix}(\lambda'\mu') \\ \mathrm{H}'\end{matrix} \right\rangle^2} . \end{aligned} \tag{4.4}$$

The spin and isospin indices in the numerator of this formula are omitted for brevity.

Let the state $|(n0)^A, [4^k], (\lambda\mu)\mathrm{H}\rangle$ correspond to a normal occupation of single-particle orbits and be described by a single Slater determinant (3.14). Then the numerator $I(\lambda'\mu')$ in (4.4) is non-vanishing only if the orbit $(n'_x n'_y n'_z)$ is empty in the state $|(\lambda\mu)\mathrm{H}\rangle$. This requirement, being due to the Pauli principle, limits the number of nonvanishing eigenvalues $r_{(\lambda'\mu')}$, namely, $r_{(\lambda'\mu')} \neq 0$, if the corresponding IR $(\lambda'\mu')$ in the expression (4.3) is connected with the single-particle orbit $(n'_x n'_y n'_z)$ which is empty in the state $|(\lambda\mu)\mathrm{H}\rangle$ of the even-even nucleus under consideration.

For calculating $I(\lambda'\mu')$ we shall use the projection operator (PO) introduced in refs. [13-15]):

$$P^{(\lambda'\mu')}_{\mathrm{H'H'}} = |(\lambda'\mu')\mathrm{H}'\rangle\langle\mathrm{H}'(\lambda'\mu')| = P_{xy}P_{zy}P_{zx}, \tag{4.5}$$

where

$$P_{zx} = \sum_{r_1} (-1)^{r_1} \frac{(\lambda'+1)!}{r_1!(\lambda'+r_1+1)!} A^{r_1}_{xz} A^{r_1}_{zx},$$

$$P_{xy} = \sum_{r_2} (-1)^{r_2} \frac{(\mu'+1)!}{r_2!(\mu'+r_2+1)!} A^{r_2}_{yx} A^{r_2}_{xy},$$

$$P_{zy} = \sum_{r_3} (-1)^{r_3} \frac{(\lambda'+\mu'+2)!}{r_3!(\lambda'+\mu'+r_3+2)!} A^{r_3}_{yz} A^{r_3}_{zy}, \tag{4.6}$$

and the generators A_{ij} are determined by the expression (3.7). In terms of the PO (4.5) the numerator of eq. (4.4) can be rewritten as follows:

$$I(\lambda'\mu') = \langle (n0)^A, [4^k], (\lambda\mu)\mathrm{H}| a_{n'_x n'_y n'_z} P^{(\lambda'\mu')}_{\mathrm{H'H'}} a^+_{n'_x n'_y n'_z} |(n0)^A, [4^k], (\lambda\mu)\mathrm{H}\rangle, \tag{4.7}$$

i.e. it is enough just to project the Slater determinant on the state with definite values of λ' and μ', since the correct values of quantum numbers $[4^k 1]$, $S'M'_S$, $T'M'_T$ for the $A+1$ nucleon intermediate states are guaranteed automatically.

Though, formally, the expressions (4.6) are infinite power series, the powers r_1, r_2, r_3 are limited positively in the matrix element (4.7). In particular, if the orbit $(n'_x n'_y n'_z)$ in (4.7) corresponds to the $A+1$ nucleon normal occupation, the vector $a^+_{n'_x n'_y n'_z}|(\lambda\mu)\mathrm{H}\rangle$ is the highest one itself. Therefore, only the terms with $r_1 = r_2 = r_3 = 0$ in (4.6) give a nonvanishing contribution to (4.7), i.e. the PO is equal to unity and $I(\lambda'\mu') = 1$. In this case $r_{(\lambda'\mu')}$ equals the SU(3) group inverse quadrant of the Clebsch-Gordan coefficient $J(\lambda'\mu')$. Thus, the GDM eigenvalues can exceed 1.

General formulae for the calculation of $J(\lambda'\mu')$, (B.1), and $I(\lambda'\mu')$, (B.3), (B.6), (B.7), are obtained in appendix B. These equations allow us to find the nonvanishing GDM eigenvalues for any arbitrary SU(3) multiplet $(\lambda\mu)$ corresponding to the normal occupation of the oscillator orbits in the n-shell. The simplest is the case of the SU(3) multiplet $(\lambda 0)$ when, according to eq. (4.3), $n'_y = 0$, i.e. the orbit of an odd nucleon belongs to the first row of the triangle (3.13). In this case, the nonvanish-

ing GDM eigenvalues can be expressed in terms of the finite series

$$\rho(abcd) = \frac{(a+b+1)!(a-c)!c!}{a!b!} \sum_{r=0}^{c-d} \frac{(-1)^r(b+r)!}{r!(a+b-c+r+1)!(c-r)!} \tag{4.8}$$

as follows:

$$r_{(\lambda'\mu')} = \rho(\lambda, n'_z, n'_x, n_{xr}) . \tag{4.9}$$

Here n_{xr} is a number of oscillator quanta along the x-axis in the extreme left orbit in the first row of the triangle (3.13), which is not occupied by the A nucleons. For example, in the case of $n=2$, $A=4$, $(\lambda\mu)=(80)$ (^{20}Ne) only the first orbit in the first row is occupied. Therefore, $n_{xr}=1$ and eq. (4.9) gives the results $r_{(10\,0)}=0$, $r_{(81)}=\frac{5}{4}$, $r_{(62)}=\frac{27}{28}$ in agreement with the calculation of Janssen [4]).

As an illustration of a more complex situation, all the eigenvalues of the GDM for nuclei with 48 nucleons in the $n=6$ shell are presented in fig. 1. It is clear from this figure that the diagonal form of the GDM contains a set of vanishing elements (the number of such elements is equal to that of occupied orbits in the shell) and the non-zero elements are close to 1.

4.1. ASYMPTOTIC ESTIMATION OF THE GDM EIGENVALUES

Let us calculate the eigenvalues of the GDM in the case of a large nucleon number A. For this aim the expression (4.7) for the numerator of (4.4) is valid. It can be

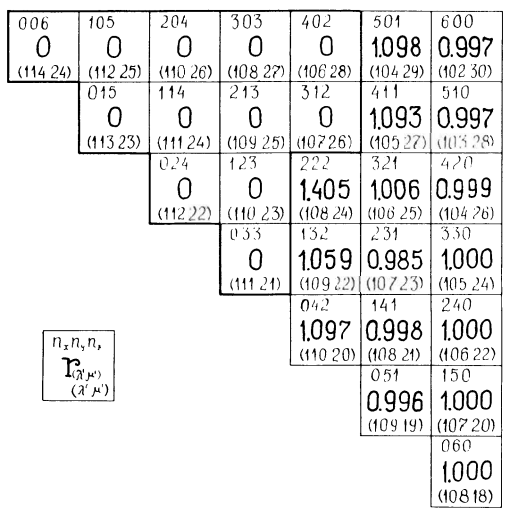

Fig. 1. GDM eigenvalue table for the ground SU(3) multiplet $(\lambda\mu)=(108\,24)$ in the case $n=6$ and $A=48$. (The occupied orbits are enclosed with a thick line.) Each square corresponds to a definite oscillator orbit $(n'_x n'_y n'_z)$. These quantum numbers are shown in the upper part of each square. In the lower part the $(\lambda'\mu')$ value is given ($\lambda'=\lambda+n'_z-n'_x$, $\mu'=\mu+n'_x-n'_y$). The GDM eigenvalue $r_{(\lambda'\mu')}$ for this SU(3) multiplet is written in the centre of each square.

written in the form

$$I(\lambda'\mu') = \sum_{\Lambda,\Lambda'} \langle(\lambda\mu)\mathrm{H}|a_{n_x'n_y'n_z'}P_{xy}|\Lambda\rangle\langle\Lambda|P_{zy}|\Lambda'\rangle\langle\Lambda'|P_{zx}a^+_{n_x'n_y'n_z'}|(\lambda\mu)\mathrm{H}\rangle\,. \quad (4.10)$$

The summation over all the intermediate states Λ and Λ' must be performed including the state $a^+_{n_x'n_y'n_z'}|(\lambda\mu)\mathrm{H}\rangle$. These states have a 1p (one-particle) or 2p1h (two-particles–one-hole) structure. In fact, since the occupation of orbits by the A nucleons is normal, the raising generators in the PO, P_{xy} and P_{zx}, can affect only the orbit occupied by the $(A+1)$th particle. As for the lowering ones, which are placed to the left of the raising operators, they can affect both the orbits occupied by the A nucleons and that of the $(A+1)$th particle. However, the lowering generators cannot form more than one hole in the states occupied by the A nucleons. Otherwise, it would be impossible to obtain the resultant bra vector $a^+_{n_x'n_y'n_z'}|(\lambda\mu)\mathrm{H}\rangle$ in the formula (4.10). Therefore, the lowering and raising generators must act so that one of the nucleons from a fully occupied orbit either follows an odd-nucleon orbit or returns to its former site. Simultaneously, the odd nucleon must fill up the formed hole or return to its former site.

If the number of nucleons A in a shell is great, those terms with intermediate states $|\Lambda\rangle = |\Lambda'\rangle = a^+_{n_x'n_y'n_z'}|(\lambda\mu)\mathrm{H}\rangle$ give the main contribution to the sum (4.10). It can be proved for the types of multiplets with $\lambda \sim \mu \sim A^{3/2}$ and $\lambda \sim A^{3/2}$, $\mu \sim A$ or $A^{1/2}$ that the contribution of the remaining terms is $A^{1/2}$ times smaller than that of the main term. This assumption makes it possible to obtain for the nonvanishing GDM eigenvalues the following expression:

$$r_{(\lambda'\mu')} = r_{zx}r_{xy}r_{zy}\,. \quad (4.11)$$

The factors in this formula are expressed by the finite series (4.8):

$$r_{zx} = \rho(\lambda n_z' n_x' n_{x\mathrm{r}})\,, \qquad r_{xy} = \rho(\mu n_x' n_y' n_{y\mathrm{c}})\,, \qquad r_{zy} = \rho(\lambda+\mu+1\, n_z' n_y' n_{y\mathrm{d}})\,. \quad (4.12)$$

They depend on the numbers $(\lambda\mu)$ of the A-particle IR, the quantum numbers of the odd-nucleon orbit $(n_x'n_y'n_z')$ and the number of oscillator quanta along the x- and y-axis for the empty orbits nearest to the occupied ones in the row $(n_{x\mathrm{r}})$, the column $(n_{y\mathrm{c}})$ and the line (parallel to the hypotenuse of the triangle (3.13)) passing across the orbit $(n_x'n_y'n_z')$. Formulae (4.11) and (4.12) overestimate the eigenvalues $r_{(\lambda'\mu')}$. However, they give rather a good approximation to them not only for large values of n but even at $n=2$.

Using these formulae let us find the GDM eigenvalues in the limit $A \gg 1$. The simplest case is when the orbit $(n_x'n_y'n_z')$ corresponds to the normal occupation of orbits by the $A+1$ nucleons. Since $\lambda \gg n_x'$, n_y', n_z' at large A, the Stirling formula allows us to obtain the asymptotic expressions

$$r^{(n)}_{zx} \approx \exp\{n_x'(n_z'+1)/\lambda\}\,, \quad (4.13)$$

$$r^{(n)}_{zy} \approx \exp\{n_y'(n_z'+1)/(\lambda+\mu)\}\,. \quad (4.14)$$

If we also have $\mu \gg n'_x$, n'_y, then

$$r_{xy} \approx \exp\{(n'_x+1)n'_y/\mu\}. \tag{4.15}$$

If an odd nucleon is placed in an arbitrary orbit, the Leibniz criterion allows us to estimate the negligible remainder of the sign-variable descending series:

$$|1-r_{zx}| < \frac{1}{(n'_x-n_{xr}+1)!}\left[\frac{n'_x(n'_z+1)}{\lambda}\right]^{n'_x-n_{xr}+1} \exp\left\{\frac{n'_x(n'_z+1)}{\lambda}\right\}. \tag{4.16}$$

Similar formulae are valid for r_{xy} and r_{zy}.

Let us begin our analysis with the SU(3) multiplet discussed in subsect. 3.2, where $2s+2$ diagonal lines are totally occupied† and $\lambda \sim \mu \sim A^{3/2}$.

In order to obtain the asymptotic estimations of the nonvanishing eigenvalues $r_{(\lambda'\mu')}$, it is convenient to divide the section of the triangle (3.13) corresponding to empty orbits into several areas, as shown in fig. 2. In areas 1-4 one or two factors in the r.h.s. of eq. (4.11) are equal to unity, since one or two of the equations $n_{xr}=0$, $n_{yc}=0$, $n_{yd}=0$ are true. We assume that the area includes a boundary, too. Considering each area separately, it is possible to prove by using eqs. (4.13)-(4.16) that $|1-r_{(\lambda'\mu')}| < A^{-1/2}$.

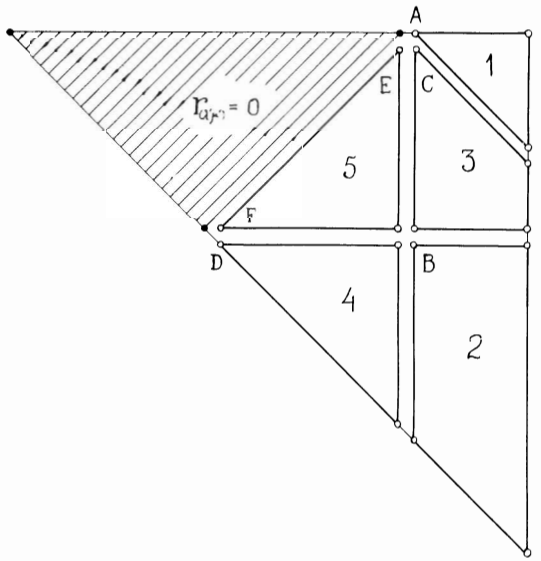

Fig. 2. The distribution of the space of empty orbits (○) into areas for the SU(3) multiplet with $\lambda \sim \mu \gg 1$ (the successive occupation of lines perpendicular to the hypotenuse of the triangle). The occupied orbits (●) are shown schematically by solid lines with the indication of their occupation directions.

† If the last diagonal line is occupied partially, the general conclusion will hold since for the multiplet under consideration the closed subshell effect is absent.

Let us proceed to a less trivial case of the SU(3) multiplet with $\lambda \gg \mu$ exhibiting a subshell structure. Setting aside for the moment the special case with $(\lambda 0)$, consider the SU(3) multiplet with $\mu \neq 0$ when the $(p+1)$th column is occupied and the $(p+2)$th one is occupied partially (the orbits with $n_y > q$ are assumed empty in this column). According to eqs. (3.16) and (3.17) we obtain $\lambda \sim A^{3/2}$. As for μ, it depends on the degree of occupation of the $(p+2)$th column: $\mu \sim A^{1/2}$ for $q \sim 1$ or p and $\mu \sim A$ for $q \sim \frac{1}{2}p$. Hence, in this case it is possible to use the asymptotic estimations of the GDM eigenvalues only for nuclei with $q \sim \frac{1}{2}p$.

Let us divide the set of empty orbits into five areas 1–5 as presented in fig. 3 and obtain the asymptotic estimations of nonvanishing $r_{(\lambda'\mu')}$ for each area individually. In area 1 existing at $2p \leq n-3$ all the eigenvalues $r_{(\lambda'\mu')} = 1$. In areas 2–4 the inequality $|1 - r_{(\lambda\mu)}| < A^{-1/2}$ is valid as can be proved. Area 5 corresponds to the lower unoccupied part of the $(p+2)$th column whose end orbits are E and F. None of the r_{ij} coincides with unity in this area. However, all the orbits admitted for the odd particle lie near the occupied ones. Therefore, $r_{(\lambda'\mu')} = r_{xy} r_{zx}^{(n)} r_{zy}^{(n)}$. Using the formulae (4.13) and (4.14) for the latter two factors we obtain

$$r_{(\lambda'\mu')} \approx r_{xy} \exp\{(2A)^{-1/2}\}. \tag{4.17}$$

This value is maximal at the point E corresponding to a normal occupation for the $A+1$ particles:

$$r_{(\lambda'\mu')}^{\max} \approx \frac{(\mu + n_x' + 1)!(\mu - n_y')!}{(\mu + n_x' - n_y' + 1)!\mu!} \exp\{(2A)^{-1/2}\}. \tag{4.18}$$

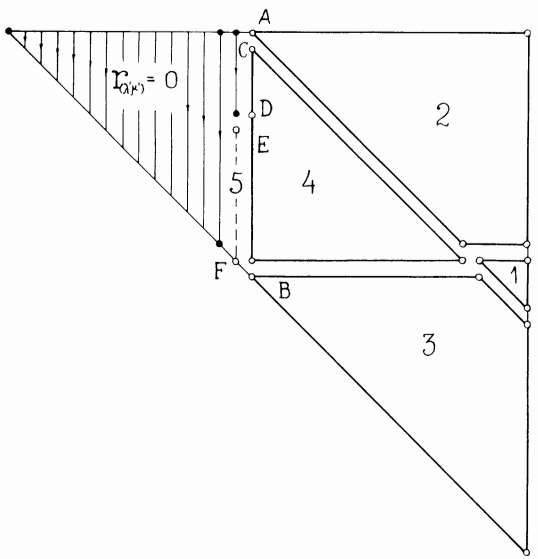

Fig. 3. The same as in fig. 2 for the ground SU(3) multiplet with $\lambda \gg \mu$ (the successive occupation of columns).

In this expression the coefficient before the exponential factor depends on the occupation degree of the $(p+2)$th column. In the limit $A \gg 1$ it increases slowly from the value $\frac{5}{4}$ when only the first upper orbit is occupied in this column $(q=0)$ up to the value $\frac{4}{3}$ when only the lowest orbit F is empty in this column $(q=p)$. If approximately half of the $(p+2)$th column is occupied $(q \sim \frac{1}{2}p)$, we obtain $r^{\max}_{(\lambda'\mu')} = \exp(\frac{1}{4}) = 1.28$. The rest of the eigenvalues can be calculated by the formulae (4.8), (4.12) and (4.17). In oscillating they approach unity rather quickly when the distance from the orbit E increases. For illustration, table 1 presents the asymptotic eigenvalues for odd-nucleon orbits in the $(p+2)$th column arranged below the orbit E. The q-number is assigned to the occupation degree of this column and $n'_y - q$ numbers the orbits in area 5 beginning with the orbit E $(n'_y - q = 1)$. Thus, the first column of table 1 shows the variation of $r^{\max}_{(\lambda'\mu')}$ depending on the $(p+2)$th column occupation. It is seen from the table that the number of the GDM eigenvalues considerably differing from unity is small in area 5. Just these eigenvalues violate the GDM normalization condition $R^2 = R$ in the case of $A \gg 1$.

In the case of the $(\lambda 0)$ ground multiplet area 5 is absent since all the subshells are filled. Therefore, all the nonvanishing eigenvalues are equal to unity in the limit $A \gg 1$ and the normalization condition for the GDM is met. Earlier this result was derived by Mazepus [16]).

The résumé of the asymptotic analysis of the GDM eigenvalues is as follows. In the case of SU(3) multiplets with $\lambda \sim \mu \sim A^{3/2}$ all the GDM eigenvalues are equal either to 0 or 1 in the limit $A \gg 1$. The normalization condition for the GDM is valid. In such multiplets the degree of collectivity for operators $\hat{Q}_0$ and $\hat{Q}_{\pm 2}$ is the same and quadrupole moments are of order $Q_0 \sim Q_{\pm 2} \sim A^{2/3}$ according to eqs. (3.12). Hence, the nonaxial parameter $\gamma \sim 1$ and the collective space represents a system of rotational levels of the nonaxial nucleus. The reduced probabilities of the E2

TABLE 1

The GDM eigenvalues $r_{(\lambda'\mu')}$ in area 5 of fig. 3 for $A \gg 1$

q	$n'_y - q$					
	1	2	3	4	5	6
0	1.250	0.937	1.016	0.996	1.001	1.000
1	1.266	0.949	1.009	0.999	1.000	1.000
2	1.271	0.954	1.007	0.999	1.000	1.000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\frac{1}{2}p$	1.284	0.963	1.003	1.000	1.000	1.000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$p-3$	1.294	0.971	1.001	1.000	-	-
$p-2$	1.298	0.973	1.001	-	-	-
$p-1$	1.306	0.980	-	-	-	-
p	1.333	-	-	-	-	-

transitions between various levels are of the same order as follows from the expression (3.11).

In the case of the ground SU(3) multiplet with $\lambda \gg \mu \gg 1$ (for nuclei with $A < 2\Omega$) the situation is more complicated. If the partially occupied subshell (column) has a few occupied or empty orbits, almost all the eigenvalues of the GDM equal 0 or 1 asymptotically, except for a few particular eigenvalues. Nevertheless, the normalization condition is not fulfilled. The degree of collectivity for the operators $\hat{Q}_0$ and $\hat{Q}_{\pm 2}$ is different since $Q_0 \sim A^{3/2}$ and $Q_{\pm 2} \sim A^{1/2} (\gamma \sim A^{-1})$ according to eqs. (3.12). The collective space is nonhomogeneous and splits into a system of rotational bands. Each of them is characterized by some definite quantum number K from the set (3.4). The intraband E2 transitions are strong (due to the $\hat{Q}_0$ operator). The interband transitions are of a single-particle order. Practically speaking, the situation remains the same in the area when approximately a half of this last subshell is occupied by nucleons. The only difference is that both operators $\hat{Q}_0$ and $\hat{Q}_{\pm 2}$ are subjected to collectivization, although to a different degree: $Q_0 \sim A^{3/2}$, $Q_{\pm 2} \sim A$. Since the nonaxial parameter $\gamma \sim A^{-1/2}$ is still small, K is a rather "good" quantum number.

Finally, in the case of the $(\lambda 0)$ multiplet the collective space degenerates into a single rotational band with $K = 0$ (the axial symmetry) characterized by strong intraband E2 transitions ($Q_0 \sim A^{3/2}$). All the eigenvalues of the GDM are equal to 0 or to 1 in the asymptotic limit and the equation $R^2 = R$ is valid in this case.

4.2. CONSISTENCY CONDITIONS

The consistency condition for a number of particles can be obtained from the general relation (2.10) connecting the number of particles A with the GDM eigenvalues $r_{(\lambda'\mu')}$. Since the degree of degeneration of the latter is equal to $4 \dim (\lambda'\mu')$ and the dimension of the collective space coincides with $\dim (\lambda\mu)$ in the Elliott model, we have the following consistency condition:

$$4\Omega - A = 4 \sum_{(\lambda'\mu')} \frac{(\lambda'+1)(\mu'+1)(\lambda'+\mu'+2)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)} r_{(\lambda'\mu')} . \tag{4.19}$$

The problem of consistency conditions for the collective operators (2.5) (i.e. for the quadrupole momentum Q_m and angular momentum L_i) is more tedious. Using the algebra of the SU(3) tensor operators it is possible to show that the consistency condition for all these operators is reduced to the single equation

$$g_2(\lambda\mu) = 2 \sum_{(\lambda'\mu')} \frac{(\lambda'+1)(\mu'+1)(\lambda'+\mu'+2)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)} [g_2(\lambda\mu) + g_2(n0) - g_2(\lambda'\mu')] r_{(\lambda'\mu')} \tag{4.20}$$

containing the eigenvalues g_2 (3.10) of the SU(3) Casimir operator. Obviously, the set of two equations (4.19) and (4.20) does not guarantee an unambiguous determination of the GDM eigenvalue in contrast to the Lipkin model [6]). Eqs. (4.19) and

(4.20) are useful in other respects. They allow us to determine the part played by GDM eigenvalues other than 0 or 1 in calculations of the collective characteristics of the system. Let us consider the asymptotic limit $A \gg 1$. According to subsect. 4.1, the number of nonvanishing eigenvalues $\boldsymbol{r}_{(\lambda'\mu')}$ other than 1 is of the order unity. On the other hand, the number of terms in the sums (4.19) and (4.20) (i.e. the number of the SU(3) multiplets $(\lambda'\mu')$ with $\boldsymbol{r}_{(\lambda'\mu')} \neq 0$) is of order n^2. Therefore, the consistency condition may be approximately satisfied provided that only the 0 or 1 figures be used as GDM eigenvalues. Simple calculations show that the consistency conditions are fulfilled in this assumption to an accuracy of order A^{-1}.

5. Conclusion

In the present paper the Elliott model concerning the occupation of an arbitrary harmonic oscillator shell by A nucleons of two sorts was discussed. Since the main quantum number n of the shell is arbitrary, the investigation of systems with a large number of particles is possible, in contrast with Janssen's paper [4]). In this case the Elliott model can describe many-body systems with various collective properties including a single rotational band (the SU(3) multiplet $(\lambda 0)$), a system of rotational bands in a prolate axial nucleus (the SU(3) multiplet $(\lambda\mu)$ with $\lambda \gg \mu$), and a rotational band of a nonaxial nucleus (the SU(3) multiplet with $\lambda \sim \mu$).

We have restricted ourselves to the most energetically favourable SU(3) multiplets only. As shown in subsect. 3.1, they correspond to many-particle configurations with a normal occupation of single-particle orbits in a shell by nucleons. Exact solutions (energy levels and eigenfunctions) were obtained for these multiplets. These have allowed us to construct the generalized density matrix (GDM) R and to find its eigenvalues $\boldsymbol{r}$ by using eqs (4.4), (B.1) and (B.6). The analysis of the GDM eigenvalues shows that R in a diagonal form contains a set of zeros. The number of zeros coincides with that of the occupied single-particle orbits. The rest of the eigenvalues $\boldsymbol{r}$ are close to unity. The important question is: How small are the deviations of nonvanishing eigenvalues $\boldsymbol{r}$ from unity? This determines the validity of the normalization condition $R^2 = R$. This question was analyzed in the asymptotic limit of a large number of particles in the system. The result of the analysis shows that the answer depends on the collective space structure. One should distinguish between two cases:

(a) the SU(3) multiplets $(\lambda\mu)$ with $\mu \ll \lambda$;

(b) the SU(3) ones $(\lambda 0)$ or $(\lambda\mu)$ with $\lambda \sim \mu$.

In the case of (a) the collective space appears nonhomogeneous splitting into a number of bands so that interband collective transitions are considerably weaker than intraband ones. In the (b) case the collective space is homogeneous and can be treated as a single collective band.

In the case of (b) all the nonvanishing eigenvalues of R are equal to unity in the asymptotical limit. In the case of (a) a small number of nonvanishing eigenvalues

differ from unity even in the asymptotical limit. In this case the normalization condition $R^2 = R$ is invalid since the space of the corresponding SU(3) multiplet is nonhomogeneous and does not satisfy the definition of the collective band given in subsect. 2.1. It should be noted that this violation of the normalization condition is different from that, due to a small number of particles in a system, noted by Janssen [4]). Is it possible to neglect this difference and to use approximate GDM eigenvalues equal to 0 or 1? As should have been expected, this approximation is valid for the evaluation of the collective quantities. Thus, if we are interested in some narrower problem of analyzing only the collective properties of nuclei, it is permissible to assume approximately that the condition $R^2 = R$ is valid even in the case of nonhomogeneous collective space. We suggest that this conclusion is a general one and not characteristic of the Elliott model only. This condition simplifies remarkably the problem in a general case when R is obtained by solving the dynamic equation (2.4). Therefore, the possibility of using it, confirmed by us for a rather sophisticated model, is a hopeful result making it easier and more realistic to solve the problem of microscopic foundation of the interacting boson model.

Appendix A

CALCULATION OF THE a^+ OPERATOR MATRIX ELEMENT

Let us evaluate the matrix element of the a^+ between the states with $A = 4k$ and $A+1$ nucleons by using many-particle wave functions (3.5). According to the Wigner-Eckart theorem for the U(4Ω) group we find

$$\begin{aligned}\langle (n0)^{A+1}, [4^k 1], (\lambda'\mu') L'M'K', \tfrac{1}{2}\sigma, \tfrac{1}{2}\tau | a^+_{lm\sigma\tau} | (n0)^A, [4^k], (\lambda\mu) LMK, S = T = 0\rangle \\ = \langle A+1 \| a^+ \| A\rangle \langle (n0)^{A+1}, [4^k 1], (\lambda'\mu') L'M'K', \tfrac{1}{2}\sigma, \tfrac{1}{2}\tau | (n0)^A, \\ \times [4^k], (\lambda\mu) LMK, S = T = 0; lm\sigma\tau\rangle = \langle A+1 \| a^+ \| A\rangle \prod_{i=1}^{7} I_i , \end{aligned} \tag{A.1}$$

since the operator a^+ transforms according to the IR [1] of the U(4Ω) group [17]) and the Clebsch-Gordan coefficient for this group is factorized into products of isofactors I_i according to the Racah lemma [18]). The isofactor U(Ω) $\supset$ SU(3)

$$I_1 = \langle (n0)^{A+1}, [4^k 1], (\lambda'\mu') | (n0)^A, [4^k], (\lambda\mu); (n0)\rangle \tag{A.2}$$

is an orbital single-particle coefficient of fractional parentage for the configuration $(n0)^{A+1}$. The spin-isospin part of this coefficient $I_2 = 1$[ref. [8])]. The isofactor SU(3) $\supset$ SO(3) is

$$I_3 = \langle (\lambda'\mu') L'K' | (\lambda\mu) LK; (n0) l\rangle . \tag{A.3}$$

Three isofactors are usual Clebsch-Gordan coefficients for the vector coupling of orbital angular momenta

$$I_4 = (LM, lm | L'M') , \tag{A.4}$$

the spin $I_5 = (00; \frac{1}{2}\sigma|\frac{1}{2}\sigma) = 1$ and isospin $I_6 = (00; \frac{1}{2}\tau|\frac{1}{2}\tau) = 1$ ones. Finally, the $U(4\Omega) \supset U(\Omega) \otimes U_{ST}(4)$ isofactor

$$I_7 = \sqrt{\frac{\dim[4^k]}{\dim[4^k 1]}} = \frac{1}{2}\sqrt{\frac{k+4}{4k+1}},$$

where $\dim[f]$ is a dimension of the IR $[f]$ of the permutation group S_A.

The reduced matrix element of the operator a^+ can be related to the number of nucleons if the expectation value of the operator $\sum_{l,m,\sigma,\tau} a^+_{lm\sigma\tau} a_{lm\sigma\tau}$ is calculated in the state (3.5). Taking into account eq. (A.1) and orthonormality of the isofactors I_i* we have

$$\langle A+1 \| a^+ \| A \rangle = \sqrt{A+1}\,. \tag{A.5}$$

Finally, the matrix element (A.1) has the form

$$\langle (n0)^{A+1}, [4^k 1], (\lambda'\mu') L'M'K', \tfrac{1}{2}\sigma, \tfrac{1}{2}\tau | a^+_{lm\sigma\tau} | (n0)^A, [4^k], (\lambda\mu) LMK, S = T = 0 \rangle$$
$$= \tfrac{1}{4}\sqrt{A+16}\, I_1 I_2 I_4\,. \tag{A.6}$$

It is convenient to use a canonical basis with the quantum numbers $N_x N_y N_z$** rather than the Elliott one in some cases. The matrix element of a^+ in this basis has the form

$$\langle (n0)^{A+1}, [4^k 1], (\lambda'\mu') N'_x N'_y N'_z, \tfrac{1}{2}\sigma, \tfrac{1}{2}\tau | a^+_{n'_x n'_y n'_z \sigma\tau} | (n0)^A, [4^k], (\lambda\mu) N_x N_y N_z, S = T = 0 \rangle$$
$$= \tfrac{1}{4}\sqrt{A+16}\, I_1 \left\langle \begin{matrix} (\lambda\mu) & (n0) \\ N_x N_y N_z & n'_x n'_y n'_z \end{matrix} \middle| \begin{matrix} (\lambda'\mu') \\ N'_x N'_y N'_z \end{matrix} \right\rangle, \tag{A.7}$$

where the last factor on the right side is the SU(3) Clebsch-Gordan coefficient in the canonical basis.

Appendix B

DERIVATION OF THE GDM EIGENVALUE FORMULA

The derivation is based on the projection operator (PO) technique [13-15]). The general formula for the Clebsch-Gordan coefficients of the SU(3) group entering into the denominator of eq. (4.4) was derived in ref. [19]):

$$J(\lambda'\mu') = \frac{\lambda!\mu!(\lambda+\mu+1)!(\lambda+n'_z-n'_x+1)!(\mu+n'_x-n'_y+1)!(\lambda+\mu+n'_z-n'_y+2)!}{(\lambda-n'_x)!(\lambda+n'_z+1)!(\mu-n'_y)!(\mu+n'_x+1)!(\lambda+\mu-n'_y+1)!(\lambda+\mu+n'_z+2)!}. \tag{B.1}$$

* K is not good as a quantum number for the SU(3) group. However, it is not essential since we can use, for example, the orthonormal basis of Vergados [19]) including quantum numbers analogous to K.

** Actually, another quantum number is necessary for labelling the canonical basis vector [5,10,20]). However, we shall drop this number, since we can use just the highest-weight vector which is unambiguously described by N_x, N_y, N_z.

In order to calculate the matrix element $I(\lambda'\mu')$ it is convenient to reduce the PO (4.5) to a normal form in which all the raising generators A_{xy}, A_{zx} and A_{zy} are to the right of the lowering ones A_{yx}, A_{xz} and A_{yz}. By using the commutation relations for these operators given in ref. [10]) the PO can be reduced to the form

$$P^{(\lambda'\mu')}_{\mathrm{H'H'}} = \sum_{s_1,s_2,s_3,s_3'} G_{(\lambda'\mu')}(s_1s_2s_3s_3')A^{s_3'}_{yz}A^{s_1'}_{yx}A^{s_2'}_{xz}A^{s_1}_{xy}A^{s_2}_{zx}A^{s_3}_{zy}\,, \tag{B.2}$$

where

$$G_{(\lambda'\mu')}(s_1s_2s_3s_3') = \frac{(-1)^{s_1+s_2+s_3'}(\lambda'+1)!(\mu'+1)!(\lambda'+\mu'+2)!}{(\lambda'+s_2+1)!(\mu'+s_1+s_3-s_3'+1)!s_3!s_3'!}$$

$$\times \sum_{s=\max(0,s_3'-s_3)}^{\min(s_1,s_2)} \frac{(s_3+s)!}{s!(s_1-s)!(s_2-s)!(s_3-s_3'+s)!(\lambda'+\mu'+s_3+s+2)!}\,, \tag{B.3}$$

$$s_1' = s_1+s_3-s_3',\ s_2' = s_2+s_3-s_3'\,. \tag{B.4}$$

After substituting the PO into the expression (4.7) and permutating the raising (lowering) generators with the operator $a^+_{n'_\zeta n'_\eta n'_\zeta}$ ($a_{n'_\zeta n'_\eta n'_\zeta}$) we obtain the following result:

$$I(\lambda'\mu') = \sum_{s_1,s_2,s_3,s_3'} G_{(\lambda'\mu')}(s_1s_2s_3s_3')\langle (n0)^A, [4^k], (\lambda\mu)\mathrm{H}|[[[a_{n'_\zeta n'_\eta n'_\zeta}, A_{yz}]^{(s_3')}, A_{yx}]^{(s_1')}, A_{xz}]^{(s_2')}$$

$$\times [A_{xy}, [A_{zx}, [A_{zy}, a^+_{n'_\zeta n'_\eta n'_\zeta}]^{(s_3')}]^{(s_2')}]^{(s_1')}|(n0)^A, [4^k], (\lambda\mu)\mathrm{H}\rangle\,, \tag{B.5}$$

where the notation

$$[A, a^+]^{(s)} = [A, [A, [\ldots, [A, a^+]\ldots]$$

introduced for the s-fold commutator of the generator A with operators a^+, $[a, A]^{(s)}$ stands for the corresponding s-fold commutator of the operators a and A. The powers s_1' and s_2' in eq. (B.5) are connected with the summation indices by the expressions (B.4). Finally, we get

$$I(\lambda'\mu') = \sum_{s_1,s_2,s_3,s_3'} F(s_1s_2s_3s_3')G_{(\lambda'\mu')}(s_1s_2s_3s_3')\,, \tag{B.6}$$

where

$$F(s_1s_2s_3s_3') = \frac{(n'_x+s_1+s_3-s_3')!n'_y!(n'_z+s_2+s_3)!}{(n'_x-s_2)!(n'_y-s_1-s_3)!n'_z!}\,. \tag{B.7}$$

The summation indices s_1, s_2, s_3 and s_3' take all values allowed by the Pauli principle. This means that the operators $a^+_{n'_\zeta n'_\eta n'_\zeta}$, $a_{n'_\zeta n'_\eta n'_\zeta}$, as well as the operators

$$a^+_{n''_\zeta n''_\eta n''_\zeta} \sim [A_{xy}, [A_{zx}, [A_{zy}, a^+_{n'_\zeta n'_\eta n'_\zeta}]^{(s_3')}]^{(s_2')}]^{(s_1')}\,,$$

$$a_{n''_\zeta n''_\eta n''_\zeta} \sim [[[a_{n'_\zeta n'_\eta n'_\zeta}, A_{yz}]^{(s_3')}, A_{yx}]^{(s_1')}, A_{xz}]^{(s_2')} \tag{B.8}$$

obtained as an intermediate result of calculation of multiple commutators (B.8), correspond to an orbit which is not occupied in the state $|(n0)^A, [4^k], (\lambda\mu)\mathrm{H}\rangle$. If at least one intermediate (or final) operator a^+ or a corresponds to an occupied single-particle orbit, the appropriate coefficient $F = 0$.

In some particular case expression (B.6) can be simplified. If, for example, the orbit $(n'_x n'_y n'_z)$ belongs to the first row of the triangle (3.13) (for the multiplet $(\lambda 0)$), we have $s_1 = s'_1 = s_3 = s'_3 = 0$, $s_2 = s'_2$ in accordance with eqs. (B.8). The expression (B.6) is reduced to a single sum and the nonvanishing eigenvalues of the GDM have the form (4.8) and (4.9).

The technique for calculating $I(\lambda'\mu')$ in the general case will be illustrated by a particular example of $A = 8$ nucleons in the $n = 2$ shell. At normal occupation, the orbits (002) and (101) are full. Therefore, the ground SU(3) multiplet is characterized by $(\lambda\mu) = (84)$. The GDM eigenvalues $r_{(104)}$ and $r_{(85)}$ are vanishing because they correspond to the odd-nucleon orbit $(n'_x n'_y n'_z)$ coinciding with one of the occupied orbits. If $(n'_x n'_y n'_z) = (011)$, (200), (110) and (020), the GDM eigenvalues $r_{(\lambda'\mu')}$ with $(\lambda'\mu') = (93)$, (66), (74) and (82), respectively, are nonvanishing. The first two values of $(\lambda'\mu')$ mentioned above correspond to the normal occupation of orbits by the $A+1$ nucleons. Therefore, $I(93) = I(66) = 1$. As for the (74) multiplet, the sum (B.6) contains, in addition to the unit addend, two nonvanishing ones corresponding to the orbits $(n''_x n''_y n''_z) = (200)$ and (011). The first one is characterized by $s_1 = 1$, $s_2 = s_3 = s'_3 = 0$, and the second one has $s_1 = s_3 = s'_3 = 0$, $s_2 = 1$. Using eqs. (B.3), (B.6) and (B.7) we obtain $I(74) = 1 - \frac{1}{3} - \frac{1}{9} = \frac{5}{9}$. In the case of the (82) multiplet the sum (B.6) includes, in addition to the unit term (for $(n''_x n''_y n''_z) = (020)$), four terms with nonvanishing indices $s_1 = 1$, $s_1 = 2$, $s_3 = 1$ and $s_3 = s'_3 = 1$ for the orbits $(n''_x n''_y n''_z) =$ (110), (200), (011) and (011), respectively. In the latter two cases it is possible to obtain the intermediate orbit (011) in different ways: either by means of one raising (A_{zy}) and one lowering (A_{yz}) generator or one raising (A_{zy}) and two lowering (A_{xz}, A_{yx}) ones. As a result, we have $I(82) = 1 - \frac{1}{2} + \frac{1}{10} - \frac{2}{13} + \frac{1}{26} = \frac{63}{130}$. The denominators $J(\lambda'\mu')$ can be calculated using eqs. (B.1). Finally, we obtain the following nonvanishing GDM eigenvalues for this nucleus: $r_{(93)} = \frac{75}{52}$, $r_{(66)} = \frac{9}{7}$, $r_{(74)} = \frac{105}{104}$ and $r_{(82)} = \frac{49}{52}$.

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