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THE GENERALIZED DENSITY MATRIX IN THE LIPKIN MODEL

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Abstract: The procedure of solving equations for the generalized density matrix method is demonstrated in the Lipkin–Meshkov–Glick model. Self-consistency and adequacy of the basic equations of the method are shown. To determine the collective spectrum of the system, we used the quasiclassical approximation with the excitation of the system determined practically in the whole variation range of the nucleon–nucleon interaction value including the transitional region. As a limiting case, the random phase approximation and its validity domain have been obtained. The equivalence of one of the equations of the method, namely the equation $[R, \boldsymbol{Q}+\frac{1}{2}\boldsymbol{\tau}]=0$, to the normalization condition of the generalized density matrix $R^2=R$ has been proved.

1. Introduction

The vibrational and rotational motions are the simplest types of collective excitations. In fact, these ideal cases are rarely realized in real nuclei. Therefore one can understand interest in more general types of collective excitations allowing one to describe the transitional cases from pure vibrations to rotation ("phase transition").

The development of a transitional model is hindered by the drastic difference of the generally accepted microscopic description of vibrations (random-phase approximation – RPA) and rotation (the cranking model). The closure of the equations of motion in the restricted collective space of nucleus states seems to be a natural generalization of these methods. The equations obtained are of the Hartree–Fock type for particle and hole amplitudes but in the broadened (single-particle and collective) space with the normalization conditions for the amplitudes [generalized Hartree–Fock approximation (GHFA) [1])]. The complexity of this method (necessity of simultaneous treatment of an even nucleus and two neighbouring odd ones) restricts its application for calculations of the realistic models. The generalized density matrix method close to GHFA deals with the closed equations for the density matrix R in the broadened space [2]). The efficiency of this method for a microscopic calculation of the corrections to rotational and vibrational spectra of deformed nuclei and for other problems has been demonstrated in a number of papers [3]). The normalization conditions (or the consistency conditions equivalent to it) for the matrix R are essential for closure of the generalized density matrix

equations. Unlike the exact (but complex) normalization conditions for the amplitudes in GHFA, the normalization condition of the generalized density matrix is approximate to the same degree as the dynamic equations. The normalization condition for R in an even nucleus can be formulated as $R^2 = R$ for a wide range of problems by analogy with the conventional density matrix.

Janssen [4]) has analysed the Elliott model by the GHFA method and found that the condition $R^2 = R$ is violated and concluded that this condition is additionally not valid in the general case. This conclusion appears to be not correct. The normalization condition of the generalized density matrix (as a closure of the equations of motion) is correct with an accuracy to terms of order Ω^{-1}, where Ω is the dimension of the single-particle space. Therefore the violation of the condition $R^2 = R$ in ref. [4]) is due to the small Ω used.

In the present paper self-consistency of the generalized density matrix method is considered basically in the same manner, as in [4]), for the Lipkin–Meshkov–Glick model [5]), that is frequently used for examining approximate methods in the many-body theory. [The most important references are given in [6]), see also ref. [7]).] Among other results, the validity of the normalization $R^2 = R$ has been demonstrated for large Ω.

Another purpose of this paper is to verify the generalized density matrix method for describing the phase transition in the simple model with the aim of subsequent generalization and application to more realistic problems. As an intermediate result, the quasiclassical solution of the Lipkin model has been obtained valid both in the region of the "vibrational" and "rotational" excitations and in the transitional region.

1.1. GENERALIZED DENSITY MATRIX

The ordinary density matrix is defined by the set of expectation values from the Fermi-operator pair $a_{1'}^{+} a_1$ over the ground state of a system. In order to treat pairing phenomena one needs to include in addition the pairs† $a_{1'}^{+} a_{\tilde{1}}^{+}$ and $a_{\tilde{1}'} a_1$, i.e. to consider the average values of the following matrix operators:

$$\langle 1|AA^{+}|1'\rangle = A_1 A_{1'}^{+}$$

$$= \begin{pmatrix} a_1 \\ a_{\tilde{1}}^{+} \end{pmatrix} (a_{1'}^{+}, a_{\tilde{1}'}) = \begin{pmatrix} a_1 a_{1'}^{+}, & a_1 a_{\tilde{1}'} \\ a_{\tilde{1}}^{+} a_{1'}^{+}, & a_{\tilde{1}}^{+} a_{\tilde{1}'} \end{pmatrix}. \tag{1.1}$$

In the method of a generalized density matrix, we consider not only the expectation values of Fermi-operator pairs in the ground and excited states but also the non-diagonal matrix elements ("the transitional density matrix"), i.e. the quantities

$$\langle c, 1|R|c', 1'\rangle = (c|A_1 A_{1'}^{+}|c'). \tag{1.2}$$

† In the following $\tilde{1}$ denotes a time-conjugated single-particle state.

The states $|c)$ are supposed to belong to "a collective band" defined by a set of collective states connected with each other by large (direct or cascade) matrix elements of collective operators. Eq. (1.2) determines the matrix R both in the single-particle space and in the collective one.

To obtain dynamical equations for R we use the splitting of the four Fermi-operator matrix elements into the products of the matrix elements of fermion pairs (analogous to the Hartree–Fock–Bogoliubov (HFB) procedure). All the states of the considered collective band $|c)$ are used as intermediate states. As a result we get the basic equation for the generalized density matrix

$$[H+h, R]=0\,. \tag{1.3}$$

Here H is the collective hamiltonian which operates only in the collective space. Its eigenvalues determine energies of the collective states of the system. The self-consistent single-particle hamiltonian h contains a functional dependence on R (linear for the two-body interaction) and depends on the single-particle and collective variables.

Eq. (1.3) and the equation for the usual density matrix in the HFB approximation have formally the same form. The difference is only in the broadening of the space. Examining the hamiltonian $H+h$ in the joint space and defining its eigenstates from the equation

$$(H+h)|\Lambda)=\mathscr{E}_\Lambda|\Lambda)\,, \tag{1.4}$$

we obtain a solution of (1.3) in the form[†]

$$R=\sum_\Lambda |\Lambda) r_\Lambda (\Lambda|\,, \tag{1.5}$$

where the occupation numbers r_Λ (the eigenvalues of the generalized density matrix) are to be determined from consistency conditions for the collective hamiltonian H or for the more simple collective operators entering in H. We consider the scheme of a solution and consistency problem in more detail using the Lipkin model.

1.2. THE LIPKIN–MESHKOV–GLICK MODEL

This model represents a system of N fermions occupying two N-fold degenerate levels separated by energy ε. The state of the particle is characterized by the quantum number τ which has the value $+1$ in the upper level and -1 in the lower one, and by quantum numbers $p=1, 2, \ldots, N$ specifying the state of each nucleon within the level. The nucleons interact with each other via the "monopole–monopole" interaction without changing quantum number p. The hamiltonian has the form

$$\mathscr{H}=\tfrac{1}{2}\varepsilon \sum_{\tau,p} \tau a^+_{p\tau} a_{p\tau} + \tfrac{1}{2} V \sum_{\tau,p,p'} a^+_{p\tau} a^+_{p'\tau} a_{p',-\tau} a_{p,-\tau}\,, \tag{1.6}$$

[†] The transition from (1.3) to (1.4) and (1.5) is a reformulation rather than a solution of the problem. Eq. (1.4) is the analog of the Hartree–Fock equation, i.e. the equation with the self-consistent hamiltonian.

where V is the strength of the interaction scattering pair of particles from one level to another.

It is convenient to rename particles and holes for the lower level and to insert the new operators

$$a_p^+ = a_{p+}^+, \qquad a_p = a_{p+},$$
$$a_{\tilde{p}}^+ = a_{p-}, \qquad a_{\tilde{p}} = a_{p-}^+.$$

The hamiltonian (1.6) can be expressed then as[†]

$$\mathcal{H} = -\tfrac{1}{2}\varepsilon \sum_p (a_p a_p^+ - a_{\tilde{p}}^+ a_{\tilde{p}}) + \frac{V}{2}\left\{\left(\sum_p a_p^+ a_{\tilde{p}}^+\right)^2 + \left(\sum_p a_{\tilde{p}} a_p\right)^2\right\}. \tag{1.7}$$

Only three combinations of the Fermi operators appear in the hamiltonian (1.7). They commute with each other as angular momentum operators and can be expressed by the matrix (1.1) in the form[††]

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{\tau} A A^+). \tag{1.8}$$

Here $\boldsymbol{\tau}$ are the Pauli matrices in the quasispin space (1.1) and Tr is taken over single-particle states p and the matrix indices $\tau = \pm 1$. In particular, the operator $\mathcal{L}_z$ is equal to half the difference in the number of particles in the upper and lower levels. Thus the maximum possible value for $\mathcal{L}_z$ and therefore for $\mathcal{L}$ is $\frac{1}{2}N$.

Inserting the angular momentum operators (1.8) into the hamiltonian (1.7) we obtain

$$\mathcal{H} = \varepsilon \mathcal{L}_z + V(\mathcal{L}_x^2 - \mathcal{L}_y^2). \tag{1.9}$$

Representation of other one-body operators is similar to eq. (1.8). In particular the particle-number operator has the form

$$\mathcal{N} = \mathrm{Tr}\,(1 - AA^+). \tag{1.10}$$

2. The method of solution

Here we demonstrate the procedure of obtaining the generalized density matrix equations and the scheme of their solution using the Lipkin model.

[†] The generalized density matrix method applied to the present problem has an accuracy of $1/N$. Therefore we omit the terms of such order in the hamiltonian (1.7).

[††] The matrix AA^+ (free of indices) is determined as

$$AA^+ = \sum_{1,2} |1\rangle A_1 A_2^+ \langle 2|.$$

Therefore the trace in eq. (1.8) is to be understood as follows:

$$\mathrm{Tr}\,(\boldsymbol{\tau} A A^+) = \sum_{p,\tau,\tau'} (A_p A_p^+)_{\tau\tau'} \boldsymbol{\tau}_{\tau'\tau}$$

where τ and τ' are the quasispin indices.

2.1. THE EQUATIONS OF MOTION AND THEIR CLOSING UP

From eqs. (1.1) and (1.8) we obtain by direct commutation

$$[\mathscr{L}, AA^+] = -[\tfrac{1}{2}\boldsymbol{\tau}, AA^+]\,. \tag{2.1}$$

Using eqs. (1.9) and (2.1) we get

$$[\mathscr{H}, AA^+] = -[\tfrac{1}{2}\varepsilon\tau^z + V(\mathscr{L}_x\tau^x - \mathscr{L}_y\tau^y), AA^+]\,. \tag{2.2}$$

The passing from (2.2) for the second-quantized operators to the equation for the density matrix is performed by calculation of the matrix elements from both sides of eq. (2.2) between the states of the "collective band" $|c)$ utilising the splitting discussed in subsect. 1.1.

The matrix element $(c|\mathscr{Q}|c)$ from the second-quantized operator $\mathscr{Q}$ between the states described by many-body functions of the whole system is a matrix over quantum numbers of the collective band. The dimension of this matrix is determined by the dimension of the collective space that is only a small part of the whole space of the system. We introduce the formal operator Q in the space $|c\rangle$, isomorphous to the subspace of the collective band $|c)$, to fulfil a correspondence

$$\langle c|Q|c'\rangle = (c|\mathscr{Q}|c')\,. \tag{2.3}$$

A particular case of this correspondence is equality (1.2) determining the generalized density matrix. If the second-quantized operator $\mathscr{Q}$ is presented in the form of the functional from AA^+ such as (1.8), the transition to the corresponding operator Q in the collective space $|c\rangle$ is performed by the formal substitution of AA^+ by R. In particular we have for the quasispin and particle-number operators:

$$\boldsymbol{L} = -\tfrac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{\tau}\boldsymbol{R})\,, \tag{2.4}$$

$$N = \mathrm{Tr}\,(1-\boldsymbol{R})\,. \tag{2.5}$$

The analog of the second-quantized hamiltonian (1.9) in the collective space is the hamiltonian H obtained with the approximate splitting of the four Fermi-operator matrix elements (see subsect. 1.1). It equals

$$H = \varepsilon L_z + V(L_x^2 - L_y^2)\,. \tag{2.6}$$

In performing a similar transition to the collective space in eq. (2.2) we obtain eq. (1.3) in which the single-particle self-consistent hamiltonian h has the form

$$h = \tfrac{1}{2}\varepsilon\tau^z + V(L_x\tau^x - L_y\tau^y)\,. \tag{2.7}$$

It follows from these expressions that h contains both the collective $\mathbf{L}$ and single-particle operators $\boldsymbol{\tau}$[†].

[†] The product of the operators in eq. (1.3) is understood as the product of the corresponding matrices in the combined (single-particle + collective) space. For example

$$hR = \sum_{c_1,2,\tau_1} \langle c\,1\tau|h|c_1 2\tau_1\rangle\langle c_1 2\tau_1|R|c'1'\tau'\rangle\,.$$

We obtain from eqs. (2.6) and (2.7)

$$H + h = \varepsilon J_z + V(J_x^2 - J_y^2)\,, \tag{2.8}$$

where the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\tau}$ is determined in the combined (single-particle + collective) space.

The hamiltonian (2.6) defining the collective spectrum of the system and the hamiltonian (2.8), allowing us to determine the density matrix of the system according to eqs. (1.4) and (1.5), have formally identical operational structure. Therefore it is sufficient to examine the symmetry and to define the eigenfunctions and eigenvalues for one of them.

2.2. THE SYMMETRY OF THE HAMILTONIAN H. QUANTUM NUMBER OF A COLLECTIVE BAND

The hamiltonian (2.6) involves only the components L_k of the angular momentum operator. It is evident that, the collective space $|c\rangle$ in this case may be constructed from the eigenvectors $|LM\rangle$ with $L = \frac{1}{2}N$ following from the definition of the operator L. The hamiltonian H does not commute with L_z but is invariant relative to rotation by 180° on the z-axis [i.e. it commutes with exp $(i\pi L_z)$]. The eigenfunctions of the hamiltonian H can be looked for in the form

$$|Ln\sigma\rangle = \sum_M |LM\rangle\langle LM|Ln\sigma\rangle\,, \tag{2.9}$$

where the quantum number $\sigma = \pm 1$ (for integer $L = \frac{1}{2}N$) are eigenvalues of the operator exp $(i\pi L_z)$ and quantum number n numbers the collective levels of the system. It is evident that the even ($\sigma = +1$) and odd ($\sigma = -1$) eigenstates expand into even and odd M, respectively. Note that hamiltonian (2.6) changes sign on transformation $\mathscr{R}_n$ involving the rotation by 180° around the bisector of the angle between axes x and y. Consequently the eigenfunctions of the hamiltonian H separate into conjugate pairs with eigenvalues different in sign. In other words the eigenvalue spectrum is symmetrical relative to $E = 0$. This property is used in the quasiclassical calculation of the collective spectrum of the system in the section below.

2.3. THE EQUATION FOR THE DENSITY MATRIX. THE CONSISTENCY CONDITIONS

By analogy with eq. (2.9) the eigenfunctions of the hamiltonian $H + h$ are sought in the form of the expansion over eigenvectors of the operator J_z

$$|J\nu\sigma\rangle = \sum_{M_J} |JM_J\rangle\langle JM_J|J\nu\sigma\rangle\,, \tag{2.10}$$

where the quantum number ν is similar to n. The functions $|JM_J\rangle$ are introduced into combined space. To separate distinctly single-particle (τ) and collective (LM)

variables we present these functions in the form

$$|JM_J\rangle = |LJM_J\rangle = \sum_{\mu} |LM_J - \mu\,; \tfrac{1}{2}\mu\rangle\langle LM_J - \mu\,; \tfrac{1}{2}\mu\,|JM_J\rangle\,. \tag{2.11}$$

Here $\langle LM\,; \frac{1}{2}\mu\,|JM_J\rangle$ is a Clebsch–Gordan coefficient, and the value $L = \frac{1}{2}N$ is a fixed parameter.

The density matrix R is expressed according to (1.5) in terms of the occupation numbers $r_{J\nu\sigma}(L)$, to determine which additional conditions are necessary. For this purpose we consider the operator identity (2.1). Calculating the matrix element from this identity between states of the collective band $|c)$ and performing the usual splitting in the left part we obtain the first condition

$$[R, \boldsymbol{L} + \tfrac{1}{2}\boldsymbol{\tau}] = 0\,, \tag{2.12}$$

that means R is a scalar relative to the momentum $\boldsymbol{J}$. The occupation numbers therefore can only depend on the quantum number J

$$r_{J\nu\sigma}(L) = r_J(L)\,.$$

Substituting this identity and expansion (2.10) in eq. (1.5) and summing over the quantum numbers ν and σ we get, using the unitary conditions of the coefficients $\langle JM_J|JL\nu\sigma\rangle$,

$$R_p = \sum_{J,M_J} |LJM_J\rangle r_J(L)\langle LJM_J|\,. \tag{2.13}$$

To determine the occupation numbers $r_J(L)$ conditions (2.5) and (2.4) are to be used, considering only one component being sufficient for the latter, since two other equations follow from the former because the matrix R is scalar. Calculating the matrix element $\langle LM|\tau^z|LJM_J\rangle$ with the functions (2.11) we obtain from (2.4)

$$-r_{L+1/2} + r_{L-1/2} = (2L+1)/2L\,. \tag{2.14}$$

The second equation is derived from the diagonal matrix elements of the particle-number operator (2.5),

$$\frac{L+1}{L} r_{L+1/2} + r_{L-1/2} = (2L+1)/2L\,. \tag{2.15}$$

The system of eqs. (2.13) and (2.14) has the following solution:

$$r_{L+1/2} = 0\,, \qquad r_{L-1/2} = \frac{2L+1}{2L}\,. \tag{2.16}$$

The accuracy of the generalized density matrix method in the Lipkin model is equal to $1/L$ (see subsect. 1.2). Thus it is not logical to retain terms of such an order in eqs. (2.14) and (2.15). In the limit of large L, it is to be regarded that $r_{L+1/2} = 0$ and $r_{L-1/2} = 1$, which is equivalent to the normalization condition of the generalized density matrix

$$R_p^2 = R_p\,. \tag{2.17}$$

It is to be noted that in calculating the generalized density matrix of the Lipkin model we did not use a definite form of the eigenfunctions of the hamiltonian $H+h$. This turned out to be possible due to the fact that R as well as the hamiltonian $H+h$ commute with the operator J^2. In the general case an explicit form of these eigenfunctions is required to determine R.

3. The collective spectrum of the Lipkin model in the quasiclassical approximation

The Schrödinger equation determining the collective spectrum of the system has the form

$$H|Ln\sigma\rangle = E_{Ln\sigma}|Ln\sigma\rangle\,, \tag{3.1}$$

where the hamiltonian H is defined by eq. (2.6). The quantity L in this equation is large since it is proportional to the particle number of the system. This enables us to use the quasiclassical approximation to determine the collective spectrum of the Lipkin model.

3.1. THE QUANTIZATION CONDITIONS AND TURNING POINTS

In passing to the quasiclassical approximation it is convenient to consider the angular momentum operators in the form [8])

$$L_x = L\zeta + \tfrac{1}{2}(1-\zeta^2)\frac{\mathrm{d}}{\mathrm{d}\zeta}\,,$$

$$L_y = -iL\zeta + \tfrac{1}{2}i(1+\zeta^2)\frac{\mathrm{d}}{\mathrm{d}\zeta}\,,$$

$$L_z = -L + \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\,,$$

where the complex variable $\zeta = x+iy$ is the stereographic projection of a point with polar angles[†] ϑ and ϕ on a sphere with radius $\frac{1}{2}$ on the plane passing through south pole:

$$\zeta = \mathrm{e}^{i\phi}\,\mathrm{ctg}\,\tfrac{1}{2}\vartheta\,.$$

Inserting these operators in the collective hamiltonian (2.6) and denoting its eigenfunctions by ψ we obtain the following differential equation in one complex variable:

$$(1+\zeta^4)\psi'' + 2\left[\frac{\varepsilon}{V}\zeta - (2L-1)\zeta^3\right]\psi' + \left[2L(2L-1)\zeta^2 - \frac{2}{V}(\varepsilon L+E)\right]\psi = 0\,. \tag{3.2}$$

[†] The angles ϑ and ϕ determine the orientation of the angular momentum vector [8]).

The variable ζ is determined in the whole complex plane and the eigenfunctions ψ must satisfy the requirement that the scalar product

$$(\psi_1\psi_2) = \int \psi_1^*(\zeta)\psi_2(\zeta)\frac{4\,\mathrm{d}^2\zeta}{(1+|\zeta|^2)^{2L+2}}$$

be finite.

We transform eq. (3.2) by substituting

$$\psi(\zeta) = U(\zeta)(1+\zeta^4)^{1/4(2L-1)}\exp\left(-\frac{\varepsilon}{2V}\operatorname{arctg}\zeta^2\right)$$

to the form

$$U''(\zeta) + (2L+1)^2 g(\epsilon, \zeta)U(\zeta) = 0\,, \tag{3.3}$$

where

$$g(\epsilon, \zeta) = \{(s-\epsilon)\zeta^4 + (1-\alpha^2-s^2)\zeta^2 - s - \epsilon\}/(1+\zeta^4)^2\,, \tag{3.4}$$

and the dimensionless value

$$\epsilon = 2E/[(2L+1)^2 V]$$

is introduced instead of the energy E. The dimensionless parameter

$$s = \frac{\varepsilon}{(2L+1)V}$$

determining a pattern of the collective spectrum of the system, varies in the interval $0 < s < \infty$.

Eq. (3.3) contains the large value $(2L+1)^2$ allowing the quasiclassical approximation to be used. The term in (3.4), containing the square of the small value

$$\alpha = \frac{1}{2L+1}$$

is not essential in this approximation and will be omitted from now on.

The quasiclassical approximation is violated in the points where the function $g(\epsilon, \zeta)$ vanishes. The zeros of this function are extremely important in the construction of the quasiclassical solution. They determine the form of the wave functions and the eigenvalue spectrum. There are four points in all in which the function (3.4) vanishes. Two of them (the points B and B′) are always located on the imaginary axis of the plane ζ symmetrical to the origin. The position of the other two points A and A′ (also symmetrical to the origin) depends on the value of the parameter s and the energy ϵ.

Consider the variation of the position of the points A and A′ in the complex plane versus the parameter s for the ground state of the system (see fig. 2). For $s > 1$ these points are on the real axis and approach the origin as s decreases. In

the transition region ($s \sim 1$) the points A and A′ are sufficiently close to each other on the real axis, if $s \leqslant 1$. As s decreases, they reach the point $\zeta = 0$, pass to the imaginary axis and begin to diverge remaining symmetrical relative to the origin. The points A and A′ are located on the imaginary axis for the parameter value $s < 1$.

The presence of four turning points produces the quasiclassical approximation in the complex plane drastically different from the conventional WKB method [9]). The quantization condition and the form of the eigenvalue spectrum are determined by the position of the "movable" turning points A and A′. (The quantization conditions are obtained in the appendix.) Accordingly one can differentiate three specific regions: "spherical" ($s > 1$) "deformed" ($s < 1$) and "transitional" ($s \sim 1$). The last is the most interesting. The value $s = 1$, as is demonstrated by Gilmore and Feng [10]), is the phase transition point for the ground state of the system with an infinite number of particles. The quasiclassical approximation is valid for the arbitrary (but sufficiently large) particle number and is characterized by the approach of the turning points in the transitional region. The quantization conditions (A.2) and (A.6) were obtained assuming the turning points to be isolated, i.e. the domains around these points, where the quasiclassical approximation is violated, do not overlap. There is an interval of the parameter s of order $\alpha^{2/3}$ (shaded in fig. 2) in the transitional region, in which the turning points A and A′ are not isolated and the quantization conditions (A.2) and (A.6) are violated.

The approach of two turning points is not an insuperable difficulty of the quasiclassical approximation for in this case the parabolic cylinder functions can be used. We restrict ourselves to the ordinary WKB method, which is a fair approximation in the transitional region, for the above-mentioned interval s is sufficiently small for large particle number.

To gain an insight into the physical nature of the approaching of the points A and A′ we consider classical trajectories of the end of the angular momentum vector on a sphere of radius L. They are obtained by averaging the hamiltonian (2.6) over the angular momentum coherent state [8])

$$|L\boldsymbol{n}\rangle = \left(\frac{2L+1}{4\pi}\right)^{1/2} (\zeta + \zeta_0)^{2L} / (1 + |\zeta_0|^2)^L ,$$

$$\zeta_0 = \mathrm{e}^{-i\phi} \operatorname{tg} \tfrac{1}{2}\vartheta ,$$

where the unit vector $\boldsymbol{n}\{\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta\}$ determines the angular momentum orientation. There are two types of the classical trajectories in the Lipkin model – global trajectories passing on the whole sphere ($0 < \phi < 2\pi$) and local ones covering only a part of the sphere.

Examine the classical trajectories near the ground state of the system. If $s > 1$, then there are only global trajectories

$$\cos\vartheta = \frac{s - \sqrt{s^2 - 2\epsilon\cos 2\phi + \cos^2 2\phi}}{\cos 2\phi} .$$

The points A, A′, B and B′ are the stereographic projections of the classical turning points of these trajectories, the points A and A′ corresponding to the points b and b′ that are nearest to the pole (see fig. 1). If $s \to 1$, then the ground-state energy of the system approaches the value $-s$ and the classical trajectories approach the south pole. Since the stereographic projection of the south pole is the origin, it is clear that the zeros of the function $g(\zeta)$ must approach the point $\zeta = 0$.

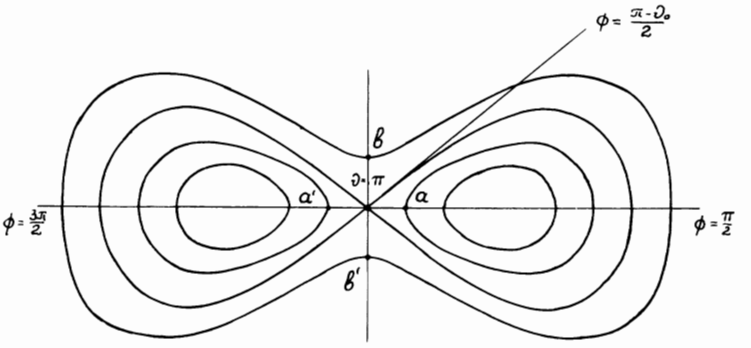

Fig. 1. The family of classical trajectories of the Lipkin model near the south pole $\vartheta = \pi$.

If $s < 1$, then the global trajectories (3.5) exist only for the energy $|\epsilon| < s$. For the energy $s < |\epsilon| < \frac{1}{2}(1 + s^2)$ two degenerate families of the local trajectories

$$\cos 2\phi = \frac{2(\epsilon - s \cos \vartheta)}{\sin^2 \vartheta} \tag{3.6}$$

appear. The centres of these trajectories are the point $\vartheta = \vartheta_0 = \arccos s$ and $\phi = 0$ in the north hemisphere, and $\vartheta = \pi - \vartheta_0$, $\phi = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$ in the south one. The points A and A′ on the imaginary axis represent the stereographic projection of the nearest to the pole classical turning points a and a′ of the local trajectories (see fig. 1). The local and global trajectories are isolated from each other by the separatrix

$$\cos \vartheta = \frac{2s + \cos 2\phi}{\cos 2\phi}$$

corresponding to the critical energy $\epsilon = -s$.

The ground-state energy of the local and global trajectories in the transitional region ($s \sim 1$) are close to the critical energy and their turning points a, a′, b and b′ to the south pole. If the ground-state energy satisfies the condition $\epsilon_0 < -s$ the mixing of the two local degenerate trajectories will take place due to tunneling through the barrier between the points a and a′. If $\epsilon_0 > -s$ then the local and global trajectories mix due to tunnelling between the points b and b′. The mixing of the classical trajectories increases the quantum fluctuations of the angular momentum vector typical of the phase transition region.

3.2. THE EXCITATIONS OF THE SYSTEM IN THE TRANSITION REGION

This region is characterized by the parameter s satisfying the condition $|1-s|<\alpha^{2/3}$. The ground-state energy of the system is close to the critical value $\epsilon=-s$. The turning point coordinates x_0 (A.1) and y_0 (A.4) in this region are therefore proportional to the small value $1-s^2$ and the expressions for the phases ϕ_0 and χ (A.3) are simplified:

$$\phi_0=\phi_{\pm}(\lambda)/q, \qquad \chi=\varphi_{\mp}(\lambda)/q,$$

where

$$\phi_{\pm}(\lambda)=\tfrac{1}{12}\sqrt[4]{1+\lambda}\{(\sqrt{1+\lambda}\mp 1)\boldsymbol{K}(k)\pm 2\boldsymbol{E}(k)\}, \qquad k^2=\frac{\sqrt{1+\lambda}-1}{2\sqrt{1+\lambda}},$$

$$\varphi_{\mp}(\lambda)=\tfrac{1}{12}\sqrt[4]{1+\lambda}\{(\sqrt{1+\lambda}\pm 1)\boldsymbol{K}(k')\mp 2\boldsymbol{E}(k')\}, \qquad k'^2=1-k^2.$$

In these formulas, $\boldsymbol{K}$ and $\boldsymbol{E}$ are the complete elliptic integrals of the first and second kind,

$$q=\alpha s/|s^2-1|^{3/2}$$

$$\lambda_n=8s(\epsilon_n+s)/(s^2-1)^2.$$

The latter quantity varies in the interval $0<\lambda_n<\infty$. The upper signs of the functions ϕ and φ refer to the case $s<1$ and the lower ones to the case $s>1$.

The quantization conditions (A.2) can be expressed as

$$\phi_{\pm}(\lambda)-\sigma q \operatorname{arctg}\left\{\tfrac{1}{2}\exp\left[-\frac{2}{q}\varphi_{\mp}(\lambda)\right]\right\}=\pi(n+\tfrac{1}{4}(2-\sigma))q. \tag{3.7}$$

In the region $s>1$ the quantum number n takes all integral values from zero to L. If $s<1$, then eq. (3.7) determined only a part of levels with the energy $\epsilon_n\geqslant -s$. Therefore $n=n_0, n_0+1, \ldots$, where n_0 is the integer

$$n_0\geqslant\frac{2}{\pi q}-\frac{\sigma}{\pi}\operatorname{arctg}\tfrac{1}{2}-\tfrac{1}{4}(2-\sigma).$$

The lower levels of the system for $s<1$ are determined by the quantization condition (A.6), which has the form in the approximation involved

$$\phi(\lambda)+\sigma q \operatorname{arctg}\left\{\tfrac{1}{2}\exp\left[-\frac{2}{q}\varphi(\lambda)\right]\right\}=\pi(n+\tfrac{1}{2})q. \tag{3.8}$$

The functions ϕ and φ entering in this equation are determined by the expressions for the phases (A.8) corresponding to the negative values of ϵ. They are equal to

$$\phi(\lambda)=\frac{1}{6\sqrt{2}}(1+\sqrt{1+\lambda})^{1/2}[\boldsymbol{E}(k)-(1-\sqrt{1+\lambda})\boldsymbol{K}(k)], \qquad k^2=\frac{2\sqrt{1+\lambda}}{1+\sqrt{1+\lambda}},$$

$$\varphi(\lambda)=\frac{1}{6\sqrt{2}}(1+\sqrt{1+\lambda})^{1/2}[\boldsymbol{E}(k')-\sqrt{1+\lambda}\boldsymbol{K}(k')], \qquad k'^2=1-k^2.$$

The root λ of eq. (3.8) is within the interval $-1<\lambda_n<0$ and the integral number n takes values from zero to n_0', where

$$n_0' \leqslant \frac{1}{\pi q}+\frac{\sigma}{\pi}\operatorname{arctg}\tfrac{1}{2}\,.$$

In general $n_0' \neq n_0$ because of violation of the quasiclassical approximation when the turning points A and A′ are approaching (see above).

The roots of eqs. (3.7) and (3.8) make it possible to determine the excitation energies

$$\omega_n = \frac{E_n - E_0}{(2L+1)V} = \frac{(s^2-1)^2}{16\alpha s}(\lambda_n - \lambda_0)\,, \tag{3.9}$$

where λ_0 is the root corresponding to the ground state with energy E_0 and $n=0$. The collective spectrum of the system is neither vibrational nor rotational. The expectation values of all the collective operators L_k are non-zero in the ground state of the system. The exact and approximate [eqs. (3.7)–(3.9)] dependence of the energy of two lowest excitations on the value $(1-s)/\alpha^{2/3}$ are shown in fig. 2. As could be expected, the accuracy of the quasiclassical approximation decreases in the region where the turning points are approaching.

Define the requirement that the points A and A′ are isolated. To this purpose we consider the circles in the complex plane with centres in these points. The circle radius is determined by the condition of application of the quasiclassical approximation in the vicinity of the zero of $g(\zeta)$:

$$\alpha|g'(\zeta)| = 2|g(\zeta)|^{3/2}\,.$$

The points A and A′ will be isolated if this radius is less than the coordinates x_0 or y_1. Hence we obtain the condition

$$\sqrt[4]{1+\lambda_n}(\sqrt{1+\lambda_n}\mp 1) > \sqrt{2}\,q \tag{3.10}$$

that these points are isolated in the state with energy $\epsilon_n > -s$ for $s<1$ (upper sign) and $s>1$ (lower sign). The region where the turning points are approaching is determined by inequality (3.10) and eq. (3.7). If $s \geqslant 1$, then the turning points A and A′ are located on the real axis and are always isolated. If $s<1$, then these points may not be isolated. This occurs when $s<1-1.16\alpha^{2/3}$ for the ground state of the system. As s further decreases the points A and A′ travel along the imaginary axis ($\epsilon_0<-s$) and begin to move apart. They become again isolated for $s<1-2.43\alpha^{2/3}$ in the ground state. The latter condition is obtained from the inequality

$$\sqrt[4]{1+\lambda_n}(1-\sqrt{1+\lambda_n}) > \sqrt{2}\,q$$

and eq. (3.8).

One can show that the turning points B and B′ on the imaginary axis and the points A and B (or A′ and B′) are isolated for all the values of the parameters s.

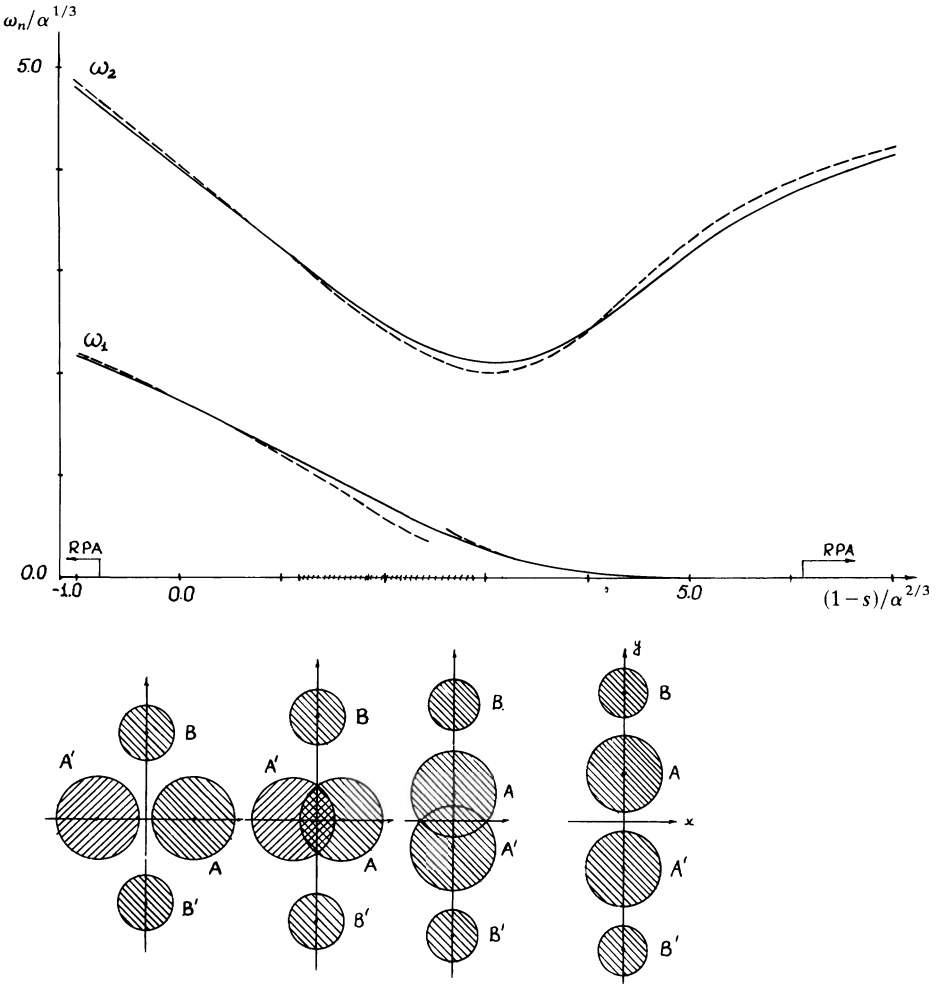

Fig. 2. The lowest excitation energy of the Lipkin model in the phase transition region for $N = 50$. The solid curves are exact solutions, the dashed curves are quasiclassical approximations. The shaded region on the ordinate axis corresponds to the non-isolated turning points A and A′ for the ground state. In the lower part of the figure, the approach of the turning points and their non-quasiclassic domains (shaded) is illustrated.

The motion of the turning points on the complex plane ζ as s decreases in the transition region is shown in fig. 2. The same figure indicates the application area of the RPA.

3.3. THE EXCITATIONS OF THE SYSTEM IN THE REGIONS DISTANT FROM THE PHASE TRANSITION POINT. THE LIMIT OF RPA

The regions remote from the transition region are determined by the values of the parameter s satisfying the inequality $|1-s| \gg \alpha^{2/3}$. In these regions, the points

A and A′ are isolated, and exponential terms in the quantization conditions (A.2) and (A.6) are small. The form of the collective spectrum of the system depends on whether the value of the parameter s is more or less than unity.

In the region of the weak nucleon–nucleon interaction ($s>1$) the exponential terms in the quantization condition (A.2) can be neglected. In this approximation, the energy of the collective levels is determined by the equation

$$(2L+1)\int_0^{y_0}[(\epsilon_n-s)y^4+(1-s^2)y^2+\epsilon_n+s]^{1/2}\frac{dy}{1+y^4}$$
$$=\pi(n+\tfrac{1}{4}(2-\sigma)), \qquad n=0,1,\ldots,L, \tag{3.11}$$

in which the coordinate y_0 is defined by the expression (A.4).

The levels constitute a roughly equidistant spectrum. The vibration frequency $\omega_1=(E_{0-}-E_{0+})/[(2L+1)V]$ determined from (3.11) has the form

$$\omega_1=\frac{\pi}{2\boldsymbol{K}(k)}[(1+s^2)^2-4\epsilon_0^2]^{1/4}, \qquad k^2=\tfrac{1}{2}\left\{1-\frac{s^2-1}{\sqrt{(1+s^2)^2-4\epsilon_0^2}}\right\}, \tag{3.12}$$

where ϵ_0 is the ground-state energy defined by (3.11). Formally, the application area of this approximation is $s-1\gg\alpha^{2/3}$. Nevertheless, it provides a good description of the first excitation of the system (the energy of the second excitation is $\omega_2=2\omega_1$ in the harmonic approximation) even for $s<1$ (see fig. 3). Formula (3.1) becomes inadequate in the region where the turning points are approaching and the frequency ω_1 vanishes for $s=1-(3\pi\alpha/\sqrt{32})^{2/3}$. Expression (3.12) turns into the frequency of RPA:

$$\omega_1^{\mathrm{RPA}}=\sqrt{s^2-1}$$

in the limit $\epsilon_0=-s$. The application area of this approximation is obtained from eq. (3.11) in case of $k^2\ll 1$ and defined the inequality

$$s\gg 1+(\sqrt{\tfrac{1}{2}}\alpha)^{2/3}.$$

The vibrations represent a deviation from the "spherical" average field with $\langle L_x\rangle=\langle L_y\rangle=0$ in the ground state of the system. The expectation values of the operator L_z determine the occupation numbers of the single-particle states $n_{p\sigma}$. The latter can be obtained from eq. (1.1) using the eigenvalues of the generalized density matrix (2.16). The occupation numbers in an arbitrarily collective state L, n, σ have the form

$$n_{p\sigma}=\tfrac{1}{2}(1+\sigma\langle Ln\sigma|L_z|Ln\sigma\rangle/L). \tag{3.13}$$

Provided the dimensionless energies ϵ_n (and therefore the collective energy $E_{Ln\sigma}$) are known the following is obtained directly from the evident consequence of eq. (2.6):

$$\langle Ln\sigma|L_z|Ln\sigma\rangle=\frac{\partial E_{Ln\sigma}}{\partial\varepsilon} \tag{3.14}$$

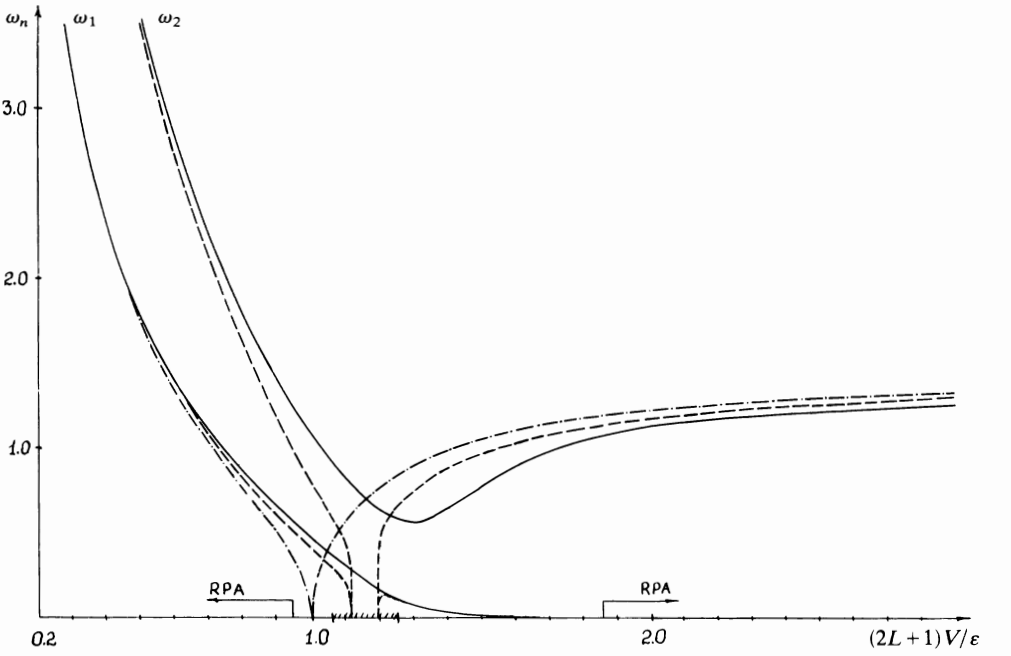

Fig. 3. The excitation energy of the Lipkin model as a function of the parameter $s^{-1} = (2L+1)V/\varepsilon$ for $N = 50$. The solid curves are exact solutions, the dashed curves are quasiclassical approximations and the dot-dashed curve is RPA. The shaded region on the ordinate axis corresponds to the non-isolated turning points A and A′ for the ground state.

Eqs. (3.13) and (3.14) determine the occupation numbers $n_{p\sigma}$ in the quasiclassical approximation. In particular, in the region where the RPA is valid we have $n_{p+} = 0$, $n_{p-} = 1$. Therefore, all the low single-particle states of the system are occupied.

In the region of the strong nucleon–nucleon interaction the system is "deformed" ($\langle L_y \rangle \neq 0$) in the ground state. Along with the vibrational spectrum representing the oscillation of the average field, there are excitations with substantially lower energy corresponding to the odd–even transitions. These excitations are the analog of the rotational spectrum.

The vibrational spectrum of the system for $\epsilon_n < -s$ is determined from the equation

$$(2L+1)\int_{y_1}^{y_2} [(\epsilon_n - s)y^4 + (1-s^2)y^2 + \epsilon_n + s]^{1/2} \frac{dy}{1+y^4} = \pi(n+\tfrac{1}{2}), \qquad n = 0, 1, 2, \ldots, \tag{3.15}$$

where y_1 and y_2 are the turning points (A.9). This equation is obtained from the quantization condition (A.6), small exponential terms being neglected. The vibra-

tional frequency can be determined from eq. (3.15)

$$\omega_2 = \frac{E_1 - E_0}{(2L+1)V} = \frac{\pi}{\sqrt{2}\,\boldsymbol{K}(k)} [\sqrt{(1+s^2)^2 - 4\epsilon_0^2} + 1 - s^2]^{1/2}, \tag{3.16}$$

$$k^{-2} = \frac{1}{2}\left\{1 + \frac{1-s^2}{\sqrt{(1+s^2)^2 - 4\epsilon_0^2}}\right\}.$$

Expression (3.16) turns into the frequency of the RPA

$$\omega_2^{\mathrm{RPA}} = \sqrt{2(1-s^2)}$$

in the limit $\epsilon_0 = -\frac{1}{2}(1+s^2)$. The region where the RPA is valid is determined by

$$s \ll 1 - 4(2\alpha)^{2/3}.$$

In this region, the single-particle occupation numbers calculated with eqs. (3.13) and (3.14) are

$$n_{p+} = \tfrac{1}{2}(1-s), \qquad n_{p-} = \tfrac{1}{2}(1+s).$$

Eq. (3.16) determines the energy of the second excited state. The lowest excitation energy equal to the energy difference of the odd and even states with $n = 0$ is proportional to the exponential terms in the right-hand side of quantization condition (A.6). It is related to the frequency ω_2 as follows:

$$\omega_1 = \frac{2}{\pi}\omega_2(\epsilon_0)\ \mathrm{arctg}\ \{\tfrac{1}{2}[1 - \exp(-\pi(2L+1)s)]\exp[-2\chi_1(\epsilon_0)]\}, \tag{3.17}$$

where the phase χ_1 is determined by eq. (A.8). If in eq. (3.17) the ground-state energy ϵ_0 is replaced by ϵ_n, the quantity ω_1 will define the odd-even splitting of the state with the quantum number n. The energies ϵ_0 and ϵ_n are found from eq. (3.15).

4. Conclusion

The microscopic description of the transitional nuclei requires more sophisticated methods solving many-body problems. In this paper we considered one of these methods, the generalized density matrix method, whose basic equations can be symbolically written as follows

$$[H + h, R] = 0,$$

$$[R, \boldsymbol{Q} + \tfrac{1}{2}\boldsymbol{\tau}] = 0.$$

Here $\boldsymbol{Q}$ is the collective operator connected with a single-particle analog $\frac{1}{2}\tau$ by a formula of type (2.4). The aim of the study was to demonstrate the consistency and adequacy of these equations for the simple model that does possess nevertheless some properties of the transitional nuclei.

The method of solving the problem actually involves three stages:

(a) the determination of the collective motion with the hamiltonian H containing a small number of collective operators (in our case $\boldsymbol{L}$);

(b) the solution of the Schrödinger equation with the hamiltonian $H+h$, describing a particle coupled with known collective motion, and expressing the generalized density matrix through the collective variables;

(c) the self-consistency procedure, i.e. expressing the collective operators through the generalized density matrix eqs. (2.4) and (2.5).

The stage (a) in the model under consideration was solved with the quasiclassical approximation. This assumption turned out to be a very convenient method in determining the collective spectrum of the system. The quasiclassical approximation in the Lipkin model allows this spectrum to be determined in a wide range of the parameter $s=\varepsilon/[(2L+1)V]$ except for the small vicinity of the phase transition point with a width of order $N^{-2/3}$. The accuracy of the quasiclassical calculations for particle number $N=50$ is no less than 1%, and in regions distant from the phase transition point it is much better.

Stage (b) for the Lipkin model is trivial since the hamiltonian $H+h$ is equivalent to H. This allows us to find the analytical solution of the consistency conditions [stage (c)].

The solving scheme essentially remains valid for more realistic models. Therefore the methods developed in this paper and the results obtained may prove to be useful in more general cases.

Appendix

DERIVATION OF THE QUANTIZATION CONDITIONS FOR THE COLLECTIVE HAMILTONIAN

For solving eq. (3.2) it is necessary to construct the quasiclassical approximation in the complex plane. The problem is simplified if only the eigenvalue spectrum of the system is to be determined. In this case it is sufficient to find the quasiclassical solution on the real axis. The energy levels are determined from the quantization conditions which can be obtained by continuing even and odd quasiclassical solutions along the positive part of the real axis from the origin to infinity. Parity conservation of the solution under continuation through all the turning points is equivalent to a boundary condition.

In the region $s>1$, where $|\epsilon|<s^{\dagger}$, there is a single turning point $A(x_0, 0)$,

$$x_0=\left[\frac{s^2-1+\sqrt{(s^2+1)^2-4\epsilon^2}}{2(s-\epsilon)}\right]^{1/2}, \tag{A.1}$$

† The boundaries of the eigenvalue spectrum depending on the parameter s are determined by the classical limit.

on the positive part of the real axis. Using the usual formulas relating the quasi-classical solutions on both sides of this point, one can obtain the following quantization conditions for even ($\sigma = +1$) and odd ($\sigma = -1$) states:

$$\operatorname{tg}\{\phi_0(\epsilon) + \tfrac{1}{4}\pi(2-\sigma)\} = \tfrac{1}{2}\sigma \exp[-2\chi(\epsilon)], \tag{A.2}$$

where

$$\begin{aligned} \phi_0(\epsilon) &= (2L+1)\int_0^{y_0} \sqrt{|g(\epsilon, iy)|}\,\mathrm{d}y, \\ \chi(\epsilon) &= (2L+1)\int_0^{x_0} \sqrt{g(\epsilon, x)}\,\mathrm{d}x, \end{aligned} \tag{A.3}$$

and

$$y_0 = \left[\frac{-s^2+1+\sqrt{(s^2+1)^2-4\epsilon^2}}{2(s-\epsilon)}\right]^{1/2} \tag{A.4}$$

is the turning point on the imaginary axis. The phase

$$\tilde{\phi}_0(\epsilon) = (2L+1)\int_{x_0}^{\infty} \sqrt{g(\epsilon, x)}\,\mathrm{d}x$$

is replaced in expression (A.2) by the phase ϕ_0 [see (A.3)] for convenience. These two values satisfy the identity

$$\phi_0(\epsilon) + \tilde{\phi}_0(\epsilon) = \pi(L+\tfrac{1}{2}),$$

which is readily proved in the complex plane.

In the region $s < 1$, where $|\epsilon| < \frac{1}{2}(1+s^2)$, there are two possibilities. If $|\epsilon| < s$, then one turning point x_0 is found on the real axis. The quantization conditions and the phases are determined by expressions (A.2) and (A.3). If $s < \varepsilon < \frac{1}{2}(1+s^2)$ then two turning points $A(x_1, 0)$ and $B(x_2, 0)$,

$$x_{1,2} = \left[\frac{1-s^2 \pm \sqrt{(1+s^2)-4\epsilon^2}}{2(\epsilon - s)}\right]^{1/2}, \tag{A.5}$$

are placed on the real axis. Continuing solutions through these points we are led to the following quantization conditions:

$$\operatorname{ctg}\phi_1(\epsilon) = 2\sigma\{\exp[-2\chi_1(\epsilon)] + \exp[-2\chi_2(\epsilon)]\}/\{4 - \exp[-2\chi_1(\epsilon) - 2\chi_2(\epsilon)]\}, \tag{A.6}$$

where

$$\begin{aligned} \phi_1(\epsilon) &= (2L+1)\int_{x_2}^{x_1} \sqrt{g(\epsilon, x)}\,\mathrm{d}x, \\ \chi_1(\epsilon) &= (2L+1)\int_0^{x_2} \sqrt{|g(\epsilon, x)|}\,\mathrm{d}x, \\ \chi_1(\epsilon) - \chi_2(\epsilon) &= \pi s(L+\tfrac{1}{2}). \end{aligned} \tag{A.7}$$

The quantization conditions for states with energy $-\frac{1}{2}(1+s^2)<\epsilon<-s$ can be obtained using symmetry of the hamiltonian relative to the transformation $\mathcal{R}_n$ (see subsect. 2.2). The states with negative energy are determined from eq. (A.6), in which ϵ is to be replaced by $-\epsilon$. As a result, the phases with altered sign of ϵ

$$\phi_1(-\epsilon)=(2L+1)\int_{y_1}^{y_2}\sqrt{|g(\epsilon, iy)|}\,\mathrm{d}y\,,$$

$$\chi_1(-\epsilon)=(2L+1)\int_{0}^{y_1}\sqrt{g(\epsilon, iy)}\,\mathrm{d}y\,, \tag{A.8}$$

enter into quantization condition (A.6). Coordinates of the turning points on the real ($x_{1,2}$) and imaginary ($y_{1,2}$) axes are connected by the identity

$$y_{1,2}(\epsilon)=\frac{1}{x_{1,2}(-\epsilon)}\,. \tag{A.9}$$

Note added in proof: After the present paper had been sent to be published the work of Klein and Li [11]) became available. In this work the quasiclassical approximation is used to determine the collective spectrum of the Lipkin model. The authors are led to the differential equation after utilizing the Holstein–Primakoff transformation for the operators L_k rather than the one-dimensional complex realization of the SU(2) group [see ref. [8])] employed in this paper. As a result Klein and Li determined only the vibrational part of the excitation spectrum away from the phase transition point. The limit of the RPA and the excitations in the transition region have not been considered.

References

1) A.K. Kerman and A. Klein, Phys. Lett. **1** (1962) 185; Phys. Rev. **132** (1963) 1326
2) S.T. Belyaev, in Development and borderlines of nuclear physics (Academic Press, NY, 1972) p. 60, and in Nuclear structure, Proc. Alushta int. School on nuclear structure (JINR, Dubna, 1972) p. 491
S.T. Belyaev and V.G. Zelevinsky, Yad. Fiz. **16** (1972) 1195; **17** (1973) 525 (Sov. J. Nucl. Phys. **16** (1973) 657; **17** (1973) 269)
3) V.G. Zelevinsky and M.I. Stockman, Izv. Akad. Nauk SSSR (ser. fiz.) **36** (1972) 2577; preprint INP 88-73, Novosibirsk, 1973; M.I. Stockman, Yad. Fiz. **22** (1975) 479 (Sov. J. Nucl. Phys. **22** (1975) 247);
V.G. Zelevinsky, Nucl. Phys. **A337** (1980) 40; **A344** (1980) 109
4) D. Janssen, Nucl. Phys. **A331** (1979) 311
5) H. Lipkin, N. Meshkov and A.J. Glick, Nucl. Phys. **62** (1965) 188
6) K.-K. Kan, P.C. Lichtner, M. Dworzecka and J.J. Griffin, Phys. Rev. **C21** (1980) 1098
7) G.J. Dreiss and A. Klein, Nucl. Phys. **A139** (1968) 81
8) I.M. Pavlichenkov, Yad. Fiz. **33** (1981) 98 (Sov. J. Nucl. Phys. **33** (1981) 52)
9) I.M. Pavlichenkov, Kurchatov Atomic Energy Institute, preprint, IAE-2769 (Moskov, 1977)
10) R. Gilmore and D.H. Feng, Nucl. Phys. **A301** (1978) 189
11) A. Klein and C.T. Li, Phys. Rev. Lett. **46** (1981) 895