EFFECTIVE TWO BODY FORCES IN NUCLEI AND GAUGE INVARIANCE

S. T. BELYAEV
Institute of Nuclear Physics, Novosibirsk, USSR

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Consequence of the conservation of nucleus current, considering it to be of one-particle structure, are discussed.

In microscopic calculations of nuclear structure there is a need to deal with the effective nucleon interaction I inside the nucleus. Lack of fundamental theory compels us to be content with a phenomenological description (parametrization) of I. Under the circumstances it is very desirable to draw a certain limitation on the possible form of I that follows from first principles. We shall consider here the consequence of the conservation of nucleon current $j_\mu = j_\mu^{(n)} + j_\mu^{(p)}$ assuming its one-particle structure. The absence of two- (and higher) particle terms in the operator j_μ is equivalent to the assumption of the gauge invariance of nuclear forces *.

Let us consider the "three point" Green function

$$K_\mu(x, y; z) = \langle T\{j_\mu(z)\Psi(x)\Psi^+(y)\}\rangle . \qquad (1)$$

In the case of current conservation ($\partial_\mu j_\mu = 0$) one obtains from eq. (1) [1]

$$\frac{\partial}{\partial z_\mu} K_\mu(x, y; z) = \langle T\{[j_0(z), \Psi(x)]\Psi^+(y)\}\rangle \delta(x_0 - z_0) + \langle T\{\Psi(x)[j_0(z), \Psi^+(y)]\}\rangle \delta(z_0 - y_0) \qquad (2)$$

which leads (by applying $j_0 = \Psi^+\Psi$) to Ward's identity

$$\frac{\partial}{\partial z_\mu} K_\mu(x, y; z) = -iG(x, y)\{\delta^4(x - z) - \delta^4(z - y)\} \qquad (3)$$

where the single-particle Green function is introduced

$$G(x, y) = -i\langle T\{\Psi(x)\Psi^+(y)\}\rangle . \qquad (4)$$

On the other hand if $j_\mu(z)$ is of one-particle structure (a bilinear combination of Ψ^+ and Ψ) then eq. (1) is a particular example of a two-particle Green function and must satisfy the corresponding Dyson equation ‡

$$K_\mu(\alpha, \alpha'; z) = K_\mu^0 - iG(\alpha, \alpha_1)\langle\alpha_1\beta_1|I|\beta_2\alpha_2\rangle \times G(\alpha_2, \alpha')K_\mu(\beta_2, \beta_1; z) \qquad (5)$$

where I is an irreducible four-pole vertex which in fact plays the role of an effective interaction. The usual way to make eq. (5) self-contained is a direct parametrization of I ‡‡.

The variable z in eq. (5) enters only into K_μ and plays the role of an external parameter. Hence eq. (5) is valid also for 4-divergence $\partial K_\mu / \partial z_\mu$ and must be satisfied by the right-hand side of eq. (3). This is obviously impossible for an arbitrarily chosen I and the condition of compatibility of eqs. (3) and (5) lays a restriction on the possible forn of the effective interaction. Before going into further details let us generalize our considerations for the most interesting case of the systems with Cooper pairing.

For systems with pairing it is necessary to consider the Green functions conserving as well as nonconserving particle number. Thus eq. (5) transforms into a set of three equations for different functions (1) (with $\Psi\Psi^+, \Psi\Psi, \Psi^+\Psi^+$ correspondingly) which contain the effective interaction

* I.e. the invariance with respect to the transformation of all interacting particle momenta $p_\mu \to p_\mu - \partial_\mu\phi(x)$ where $\phi(x)$ is an arbitrary function in a 4-coordinate space. The spin-orbit force $(\sigma^{(1)} + \sigma^{(2)})(p^{(1)} - p^{(2)}) \times \nabla F(x^{(1)} - x^{(2)})$ (which seems to be the only important velocity dependent term in the nonrelativistic region) is gauge invariant in the zero-range limit $F(x) \sim \delta(x)$.

‡ $\alpha, \beta, \ldots$ define any complete set of variables (e.g. x, x_0, spin and isospin indices). The summation (integration) over repeating indices is understood.

‡‡ This is intrinsic to the HF-method, RPA, quasiboson approximation etc., as well as to the Fermi-liquid approach.

not only in the particle-hole channel but also in the particle-particle channel. It turns out to be convenient to introduce two-component notation [2]

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \tilde{\psi}^{+}(x) \end{pmatrix}, \qquad \Psi^{+}(x) = (\psi^{+}(x), \tilde{\psi}(x)) \tag{6}$$

($\sim$ means time-conjugation) and the corresponding (2×2) unit - and Pauli - matrices $\tau^{0,1,2,3}$. Then eqs. (1), (2), (4), (5) remain formally valid, provided one considers K_μ and G as (2×2)-matrices and the effective interaction I as combination of pairs of τ-matrices of the form *

$$\begin{aligned} \langle\alpha_1\beta_1|I|\beta_2\alpha_2\rangle = {} & \langle\alpha_1\beta_1|I^{(-)}|\beta_2\alpha_2\rangle\tau^0_\alpha\tau^0_\beta + \\ & + \langle\alpha_1\beta_1|I^{(+)}|\beta_2\alpha_2\rangle\tau^3_\alpha\tau^3_\beta + \\ & + \langle\alpha_1\tilde{\alpha}_2|\bar{I}|\tilde{\beta}_1\beta_2\rangle(\tau^1_\alpha\tau^1_\beta + \tau^2_\alpha\tau^2_\beta) . \end{aligned} \tag{7}$$

The current operator then becomes

$$j_\mu(z) = \Psi^{+}(z)\hat{j}_\mu\Psi(z) + \text{const} \tag{8a}$$

$$\hat{j}_0 = \tau^3 ; \quad \hat{j}_\mu = \frac{1}{2mi}(\overrightarrow{\nabla_\mu} - \overleftarrow{\nabla_\mu})\tau^0 , \quad \mu = 1,2,3 . \tag{8b}$$

Using substitution $j_0(z)$ from eq. (8) in eq. (2) we obtain instead of eq. (3)

$$\begin{aligned} \frac{\partial}{\partial z_\mu}K_\mu(x,y;z) = {} & -i\{\tau^3 G(x,y)\delta^4(x-z) + \\ & - G(x,y)\tau^3\delta^4(z-y)\} . \end{aligned} \tag{9}$$

It is more convenient to consider a modified equation extracted from eq. (9)

$$\begin{aligned} \Lambda(x,y) \equiv {} & -i\int G^{-1}(x,x_1)\frac{\partial}{\partial z_\mu}K_\mu(x_1,y_1;z)\times \\ & \times G^{-1}(y_1,y)\phi(z)\mathrm{d}^4x_1\mathrm{d}^4y_1\mathrm{d}^4z = \\ & = \phi(x)\tau^3 G^{-1}(x,y) - G^{-1}(x,y)\tau^3\phi(y) \end{aligned} \tag{10}$$

where $\phi(z)$ is an arbitrary function of time and space coordinates. The quantity Λ defined by the first of eqs. (10) must satisfy the equation that follows from eq. (5)

$$\begin{aligned} \Lambda(\alpha,\alpha') = {} & \Lambda^0(\alpha,\alpha') + \\ & - i\,\mathrm{Tr}_\beta\langle\alpha\beta_1|I|\beta_2\alpha'\rangle G(\beta_2\beta_3)\Lambda(\beta_3,\beta_4)G(\beta_4,\beta_1) \end{aligned} \tag{11}$$

* The last term in eq. (7) describes the interaction in the particle-particle channel, the first two terms correspond to particle-hole channel and differ in T-parity with respect to one pair of variables, i.e. $\langle\alpha_1\beta_1|I^{(\pm)}|\beta_2\alpha_2\rangle = \pm\langle\alpha_1\tilde{\beta}_2|I^{(\pm)}|\tilde{\beta}_1\alpha_2\rangle$.

where Tr_β means a trace over τ_β matrix indices.

Let us find an explicit expression of Λ^0 (defined as Λ with K^0_μ instead of K_μ). According to the definition of the quantity K^0_μ we have from eqs. (1) and (8a)

$$K^0_\mu(x,y;z) = -G(x,z)\hat{j}_\mu G(z,y) .$$

Applying this together with eq. (8b) to the definition of Λ we obtain an expression analogous to eq. (10) apart from the replacement of G^{-1} by the reversed Green function of non-interacting particles

$$G^{0-1}(x,y) = \left(i\tau^0\frac{\partial}{\partial x_0} + \tau^3\frac{1}{2m}\nabla^2_x\right)\delta^4(x-y) .$$

Now we demand eq. (11) to be satisfied by Λ and Λ^0 obtained above. It gives a desirable condition for I

$$\begin{aligned} \mathrm{Tr}_\beta\langle\alpha\beta_1|I|\beta_2\alpha'\rangle\{G(\beta_2,\beta_3)\langle\beta_3|\phi|\beta_1\rangle\tau^3 + \\ - \tau^3\langle\beta_2|\phi|\beta_3\rangle G(\beta_3,\beta_1)\} = \\ = i\{\Sigma(\alpha,\alpha_1)\langle\alpha_1|\phi|\alpha'\rangle\tau^3 - \tau^3\langle\alpha|\phi|\alpha_1\rangle\Sigma(\alpha_1,\alpha')\} \end{aligned} \tag{12}$$

where the self-energy matrix Σ is introduced including an ordinary self-energy part ("self-consistent potential" U) and a "pairing potential" Δ

$$\begin{aligned} \Sigma(\alpha,\alpha') = {} & G^{0-1}(\alpha,\alpha') - G^{-1}(\alpha,\alpha') \equiv \\ & \equiv\langle\alpha|U|\alpha'\rangle\tau^3 + \langle\alpha|\Delta|\alpha'\rangle\tau^1 . \end{aligned}$$

If the effective interaction is assumed to be local in time, then eq. (12) allows a replacement of $iG(\alpha,\alpha')$ by time independent generalized density matrix

$$\begin{aligned} R(\alpha,\alpha') = {} & \langle\Psi(\alpha)\Psi^{+}(\alpha')\rangle = \\ & = \begin{pmatrix} \langle\psi(\alpha)\psi^{+}(\alpha')\rangle & \langle\psi(\alpha)\psi(\tilde{\alpha}')\rangle \\ \langle\psi^{+}(\tilde{\alpha})\psi^{+}(\alpha')\rangle & \langle\psi^{+}(\tilde{\alpha})\psi(\tilde{\alpha}')\rangle \end{pmatrix} = \\ & = \langle\psi(\alpha)\psi^{+}(\alpha')\rangle\tau^3 + \langle\psi(\alpha)\psi(\tilde{\alpha}')\rangle\tau^1 \end{aligned}$$

(where in the last equality T-parity conservation is taken into account). The matrix equation (12) is in fact equivalent to two equations in ordinary representation

$$\begin{aligned} \sum_{\{\beta\}}\langle\alpha\beta_1|I^{(-)}|\beta_2\alpha'\rangle\{\langle\psi(\beta_2)\psi^{+}(\beta_3)\rangle\langle\beta_3|\phi|\beta_1\rangle + \\ - \langle\beta_2|\phi|\beta_3\rangle\langle\psi(\beta_3)\psi^{+}(\beta_1)\rangle\} = \\ = -\sum_{\alpha_1}\{\langle\alpha|U|\alpha_1\rangle\langle\alpha_1|\phi|\alpha'\rangle - \langle\alpha|\phi|\alpha_1\rangle\langle\alpha_1|U|\alpha'\rangle\} \end{aligned} \tag{13}$$

$$\sum_{\{\beta\}} \langle \alpha \tilde{\alpha}' | \bar{I} | \tilde{\beta}_1 \beta_2 \rangle \{ \langle \psi(\beta_2) \psi(\tilde{\beta}_3) \rangle \langle \beta_3 | \phi | \beta_1 \rangle + \langle \beta_2 | \phi | \beta_3 \rangle \langle \psi(\beta_3) \psi(\tilde{\beta}_1) \rangle \} = \quad (14)$$

$$= - \sum_{\alpha_1} \{ \langle \alpha | \Delta | \alpha_1 \rangle \langle \alpha_1 | \phi | \alpha' \rangle + \langle \alpha | \phi | \alpha_1 \rangle \langle \alpha_1 | \Delta | \alpha' \rangle \} .$$

The meaning of eq. (13) is evident. The left-hand side can be rewritten in the form $\langle \psi \psi^+ \rangle (I^{(-)} \phi - \phi I^{(-)})$. Therefore eq. (13) indicates that velocity dependence in the self-consistent potential U (exhibited in noncommutation U with an arbitrary function ϕ of space-coordinates) results from the analogous dependence in T-odd particle-hole effective interaction $I^{(-)}$.

Eq. (14) is of greater fundamental nature. For the particular case $\phi \equiv 1$ it is transposed into the well-known equation for Δ

$$\langle \alpha | \Delta | \alpha' \rangle = - \sum_{\{\beta\}} \langle \alpha \tilde{\alpha}' | \bar{I} | \tilde{\beta} \beta' \rangle \langle \psi(\beta') \psi(\tilde{\beta}) \rangle . \quad (15)$$

Let the matrix elements of $\bar{I}$ which contribute into eq. (15) be given. This is sufficient for defining the density matrix and quasiparticles. Then from eq. (14) it follows that (a) the interaction $\bar{I}$ is not exhausted by the elements contributing into eq. (15), as those alone cannot satisfy eq. (14) with arbitrary $\phi(x)$, (b) additional elements of $\bar{I}$ (although not effecting single-particle spectra but generating collective modes) cannot be chosen independently. The physics of these statements is connected with a symmetry breaking on the single-particle (quasi-particle) stage. In such cases collective modes inevitably arise restoring the broken symmetry ("Goldstone theorem").

It is namely the gauge invariance that breaks on the first (single-particle) stage in the ordinary treatment of Cooper pairing. Therefore it is natural that the requirement to maintain the gauge invariance allows us to find collective modes needed for this purpose. Eq. (14) is in fact the condition of consistency between the single-particle and the collective spectra (to be more precise, between the elements of $\bar{I}$ generating those spectra).

What is the practical use of eq. (14)? If an approximation is made of $\bar{I}$ by a local potential and a strict consistency is kept during the calculation of the density matrix and Δ, then eq. (14) satisfies identically. Hence eq. (14) is useful, firstly, to verify the consistency of a solution of single-particle problem, and secondly, in the case of nonlocal interaction $\bar{I}$.

The important example of the latter is represented by the pairing interaction $\langle \alpha \tilde{\alpha}' | \bar{I} | \tilde{\beta} \beta' \rangle = - \frac{1}{2} G \delta_{\alpha \alpha'} \delta_{\beta \beta'}$, which is very convenient as it leads to $\Delta = \text{const}$ but evidently does not satisfy eq. (14). Therefore the collective spectra obtained in the model with the pairing (plus any interaction in particle-hole channel) is not complete. In this case eq. (14) allows to complete the ordinary pairing with new "interactions" which do not change the single-particle picture but restore the gauge invariance and generate new branches of collective excitations [3].

References

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