SOME ASPECTS OF COLLECTIVE PROPERTIES OF NUCLEI

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1. INTRODUCTION AND BASIC EQUATIONS

1.1. Introduction

Certain questions connected with the collective properties of nuclei will be considered in this paper. There are two different complementary approaches in the consideration of collective motions. One method - we shall call it the phenomenological one - consists of directly introducing collective degrees of freedom and the corresponding collective Hamiltonian, H_{coll}, with some phenomenological parameters. The aim of such a theory is, first, to solve the Schrödinger equation with the Hamiltonian H_{coll}, and secondly to calculate the parameters in H_{coll} under certain assumptions or using certain models. The Davydov-Filippov model can serve as an example of such an approach to the investigation of collective properties of nuclei. There it is supposed that H_{coll} describes a non-axial rotator and the parameters of the Hamiltonian are then chosen so as to obtain the best fit to experimental data.

The second possible method of investigating collective excitations of nuclei - let us call it the microscopic one - works right from the beginning with a many-body Hamiltonian $H = H_0 + H_{int}$. Then such excitations in a system of nucleons, the nature of which is collective, are studied. In this method no additional collective degrees of freedom, and hence no additional phenomenological parameters, are introduced a priori. In principle, everything is determined by the nucleon-nucleon interaction H_{int}. As a matter of fact, the interaction H_{int} must practically be introduced phenomenologically, since it has not been possible to derive it from first principles so far. The actual difference is in the "degree of phenomenologicality", i.e. in the number of parameters introduced into the theory. One needs only one or two parameters, which in principle determine the various properties of all nuclei, to determine H_{int}, whereas in the above-mentioned example of a non-axial rotator, two parameters are introduced for each nucleus (the value of the first 2 + level and the degree of non-axiality γ).

We shall deal mainly with the microscopic approach and hence the first question is that of a reasonable choice of the nucleon-nucleon interaction H_{int}. A certain form of the interaction induces certain correlations in the motion of nucleons. Thus we can draw conclusions as to H_{int}, having analysed the character of the correlations between nucleons in nuclei.

The success of the shell model indicates that a considerable part of the nucleon-nucleon interaction can be taken into account by a self-consistent field. A self-consistent field is the result of the correlation between a large number of nucleons, i.e. a long-range correlation in this sense (although it is not induced by long-range forces). Obviously some sort of residual interaction between the nucleons remains after the self-consistent field has been separated. The question naturally arises: which part of the residual inter-

action is most important? Since the self-consistent field only takes long-range correlations into account, it is only natural to suppose that the residual interaction, on the contrary, causes short-range correlations (between a few nucleons) in the limit-pairing correlations.

COOPER [1] has noticed that bound states between particles become possible near to the Fermi surface ("Cooper pairs") in macroscopic Fermi systems even in the case of weak attraction. This leads to the non-stability of the ground state of an ideal Fermi gas and to the origin of completely new properties in the system (the superconductivity of metals). Does anything analogous to Cooper pairs exist in nuclei? A positive experimental indication of this is the effect of the pairing of nucleons on one and the same level which shows itself in the mass d'fference between neighbouring even and odd nuclei. Thus it is natural to suppose that, after the self-consistent field has been separated, the basic residual interaction is such that it causes pairing correlations of the Cooper type *.

1.2. Methods for the consideration of pairing correlation

Various methods for computing pairing correlation in Fermi-systems have been developed recently. They are to a great extent equivalent, though they may be convenient to a greater or smaller degree depending on the problem under consideration. The physical idea of these methods is as follows.

Since the forming of bound pairs is "of advantage", constructing the new ground state of a system from Cooper pairs is quite natural. Let the creation operator of a pair in the bound state be

$$A^+ = \sum_{\nu\nu'} \varphi_{\nu\nu'}\, a_\nu^+\, a_{\nu'}^+ , \qquad (1.1)$$

where a_ν^+ is the creation operator of one particle in state ν and $\varphi_{\nu\nu'}$ is the wave function of the bound state of the pair. It will be quite natural, therefore, to seek the ground-state function of the system with 2N particles as

$$\Psi_0^N = (A^+)^N |0\rangle . \qquad (1.2)$$

However, in practice it is impossible to make use of such a function. If N is large, the superposition of the (1.2) functions with different N may be used instead of (1.2), for example in the form

$$\Psi_0 = e^{\sigma A^+} |0\rangle , \qquad (1.3)$$

where the parameter σ may be chosen from the condition

$$\langle \Psi_0, \hat{N} \Psi_0 \rangle = N \langle \Psi_0, \Psi_0 \rangle . \qquad (1.4)$$

Further simplification of function (1.3) is due to the assumption about pair operator (1.1). If we assume that

$$\varphi_{\nu\nu'} = \tfrac{1}{2}\, \delta_{\nu,\tilde{\nu}}\, \varphi_\nu \quad ; \quad (\varphi_{\tilde{\nu}} = -\varphi_\nu) \qquad (1.5)$$

* We are considering medium and heavy nuclei. In light nuclei correlations of the α-particle type between four nucleons may be important.

(let us call states ν and $\tilde{\nu}$ conjugated) then (1.3) may be written as

$$\Psi_0 = \exp(\sigma \sum_{|\nu|} \varphi_\nu a_\nu^+ a_{\tilde{\nu}}^+) |0\rangle = \prod_{|\nu|} (1 + \sigma \varphi_\nu a_\nu^+ a_{\tilde{\nu}}^+) |0\rangle, \quad (1.6)$$

where $|\nu|$ is the pair of states ν, $\tilde{\nu}$. A function of the (1.6) type was introduced in the paper by BARDEEN, COOPER and SCHRIEFFER on the theory of superconductivity [2]. In this case ν and $\tilde{\nu}$ correspond to states of electrons with opposite momenta and spins, and condition (1.5) means that the total momentum and spin of the bound pair is zero. A parameter $\sigma \varphi_\nu$ was defined in [2] minimizing the expectation value of the Hamiltonian of the system in state (1.6).

BOGOLYUBOV proposed quite a general and simple method for consideration of the pairing correlation [3]. The idea of this method is based on the quasi-particle concept and involves the explicit introduction of quasi-particle operators by means of canonical transformation.

A few words about the physical meaning of quasi-particles. The low-lying excited states of many-particle systems can be approximately described as a set of independent elementary excitations or quasi-particles. The ground state is then considered to be a vacuum and the excitations of the system are induced by the creation of one or several quasi-particles. Such quasi-particles in an ideal Fermi gas are holes inside the Fermi sea and particles outside it.

The form of quasi-particles in the system with pairing may be established from that of function (1.6). Let us act on (1.6) with annihilation operator a_ν:

$$a_\nu \Psi_0 = \sigma \varphi_\nu a_{\tilde{\nu}}^+ \prod_{|\nu'| \neq \nu} (1 + \sigma \varphi_{\nu'} a_{\nu'}^+ a_{\tilde{\nu}'}^+) |0\rangle = \sigma \varphi_\nu a_{\tilde{\nu}}^+ \Psi_0,$$

from which one may see that

$$(a_\nu - \sigma \varphi_\nu a_{\tilde{\nu}}^+) \Psi_0 = 0$$

and, consequently, Ψ_0 is vacuum for a combination of operators of the type $\alpha_\nu \sim a_\nu - \sigma \varphi_\nu a_{\tilde{\nu}}^+$. Thus, quasi-particle operators should be sought as

$$\alpha_\nu = u_\nu a_\nu - v_\nu a_{\tilde{\nu}}^+, \quad (1.7)$$

where one must assume $u_\nu^2 + v_\nu^2 = 1$ in order that transformation (1.7) be canonical. The equation for u_ν, v_ν will be obtained from the minimum energy condition for the ground state Ψ_0 (defined as vacuum in respect to α_ν, i.e. satisfying equations $\alpha_\nu \Psi_0 = 0$); this is equivalent to the requirement banning quasi-particle pair creation from vacuum. The method of canonical transformation allows us to abandon condition (1.5) and consider pairs of a more general type. With this instead of (1.7) it follows that

$$\alpha_\nu = \sum_f (u_{\nu f} a_f - v_{\nu f} a_f^+). \quad (1.8)$$

The calculation of average values with a function of the (1.6) type is equivalent to the independent avarage of pairs of operators, such as

$$\langle a_1^+ a_2^+ a_{2'} a_{1'} \rangle = \langle a_1^+ a_{1'} \rangle \langle a_2^+ a_{2'} \rangle - \langle a_1^+ a_{2'} \rangle \langle a_2^+ a_{1'} \rangle + \langle a_1^+ a_2^+ \rangle \langle a_{2'} a_{1'} \rangle \quad (1.9)$$

and "diagonal" expressions only

$$\langle a_1^+ a_1 \rangle = v_1^2; \quad \langle a_1^+ a_{\tilde{1}}^+ \rangle = \langle a_{\tilde{1}} a_1 \rangle = u_1 v_1 \tag{1.10}$$

being different from zero. The decomposition of averages (1.9), but without the last term, is characteristic of the Hartree-Fock method for the self-consistent field. The last term in (1.9) accounts for the Cooper pairing effect. It is easy to make sure that if a more general transformation (1.8) is used instead of (1.7), relation (1.9) still holds for averages in the α vacuum state, but apart from the diagonal values (1.10) the more general averages of any pairs of operators are different from zero.

Under the general assumption that the ground state of the system is vacuum for quasi-particles between the averages of pairs of operators there exist the relations

$$\begin{aligned} &\sum_3 (\langle a_1^+ a_3 \rangle \langle a_3^+ a_2 \rangle + \langle a_1^+ a_3^+ \rangle \langle a_3 a_2 \rangle) = \langle a_1^+ a_2 \rangle \\ &\sum_3 (\langle a_1^+ a_3 \rangle \langle a_2^+ a_3^+ \rangle + \langle a_2^+ a_3 \rangle \langle a_1^+ a_3^+ \rangle) = 0 \end{aligned} \tag{1.11}$$

The variation of the Hamiltonian average value results in the equations

$$\begin{aligned} A_{12} &\equiv \langle [a_1 a_2; H] \rangle = 0 \\ B_{12} &\equiv \langle [a_1^+ a_2; H] \rangle = 0 \end{aligned} \tag{1.12}$$

which, together with (1.11), determine $\langle a_1^+ a_2 \rangle$ and $\langle a_1 a_2 \rangle$. Equations (1.11), and (1.12) (formulated by BOGOLYUBOV [4]) are a generalization of the Hartree-Fock method for systems with Cooper pairing. We shall use this method for considering some collective properties of nuclei.

1.3. Basic equations of generalized Hartree-Fock method

Let us write down the Hamiltonian of the nucleon system in secondary-quantized representation as

$$\begin{aligned} H = &\sum_{12} (\mathcal{E}_{12} - \lambda \delta_{12}) a_1^+ a_2 \\ &+ \tfrac{1}{2} \sum_{122'1'} \langle 12 | G | 2'1' \rangle a_1^+ a_2^+ a_{2'} a_{1'} \end{aligned} \tag{1.13}$$

where $\mathcal{E}_{12}$ is the matrix of the single-particle Hamiltonian, and λ the chemical potential of the system. Let single-particle states, transient into each other by time-reversal, be designated by $(1, \tilde{1})$ or $(\nu, \tilde{\nu})$. Then, in addition to the general properties of symmetry, we have in particular

$$\mathcal{E}_{12} = \mathcal{E}_{\tilde{2}\tilde{1}}; \quad \langle 12 | G | 2'1' \rangle = \langle \tilde{1}\tilde{2} | G | \tilde{2}'\tilde{1}' \rangle^*. \tag{1.14}$$

Substituting (1.13) into the first of equations (1.12) (the second one presents nothing new) we obtain

$$\mathcal{A}_{12} = \sum_3 \{\epsilon(13)\langle a_3 a_2\rangle + \epsilon(23)\langle a_1 a_3\rangle\} + \Delta(12) \qquad (1.15)$$
$$- \sum_3 \{\Delta(13)\langle a_3^+ a_2\rangle + \Delta(32)\ \langle a_3^+ a_1\rangle\} = 0$$

where

$$\epsilon(12) = \mathcal{E}_{12} - \lambda\delta_{12} + \sum_{1'2'}\overline{\langle 11'|G|22'\rangle}\langle a_{1'}^+ a_{2'}\rangle = \epsilon^*(21) \qquad (1.16)$$

$$\Delta(12) = -\sum_{1'2'}\langle 12|G|2'1'\rangle\langle a_{2'} a_{1'}\rangle = -\Delta(21) \qquad (1.17)$$
$$\overline{\langle 11'|G|2'2\rangle} = \langle 11'|G|2'2\rangle - \langle 11'|G|22'\rangle.$$

It should be noted that the interaction enters this equation only through the Cooper pairing value Δ and single-particle Hamiltonian ϵ, where it defines the additional self-consistent field.

If we formally assume ϵ and Δ to be diagonal, i.e.

$$\epsilon(12) = \epsilon_1\delta_{12}; \qquad \Delta(12) = \Delta_1\delta_{2\tilde{1}} \qquad (1.18)$$

then equations (1.11), (1.15) will be satisfied for diagonal averages

$$\rho_{12} = \langle a_2^+ a_1\rangle = \tfrac{1}{2}\left(1 - \frac{\epsilon_1}{E_1}\right)\delta_{12}; \qquad (1.19)$$
$$\langle a_2 a_1\rangle = \frac{\Delta_1}{2E_1}\,\delta_{1\tilde{2}}$$

where

$$E_1 = \sqrt{\epsilon_1^2 + \Delta_1^2}. \qquad (1.19')$$

The quantity $\epsilon(12)$ (1.16) is the Hermitian matrix of the single-particle Hamiltonian with the additional self-consistent field, and may be diagonalized by suitable choice of the single-particle states. Physically, this means finding single-particle eigenfunctions in the total self-consistent field. Equation (1.17), however, does not lead to the diagonal expression for $\Delta(12)$, as matrix elements of the type $\langle 1\tilde{2}|G|\tilde{1}'1'\rangle$ generally speaking, are different from zero*. Nevertheless, under certain physically reasonable assumptions the condition (1.18) for $\Delta(12)$ appears to be fulfilled. This is valid, in particular, in the model of the self-consistent potential with a flat bottom, when Δ may be considered constant in space coordinate representation [6]. In any case the basic Cooper pairing effect is determined just by this value $\Delta(1\tilde{1})$, and the non-diagonal parts $\Delta(12)$, if necessary, may be considered as perturbation. For the diagonal part of Δ from (1.17) and (1.19) we obtain

$$\Delta_1 = -\sum_2\langle 1\tilde{1}|G|\tilde{2}2\rangle(\Delta_2/2E_2). \qquad (1.19'')$$

* In contrast to the case of an infinite system where they are zero because of the conservation of momentum.

For simplicity we shall in the following mostly consider Δ to be a constant, or to be more exact *

$$\Delta_1 = \gamma_1 \Delta \quad \text{where} \quad \gamma_{\tilde{1}} = -\gamma_1 ; \ |\gamma_1| = 1. \tag{1.19'''}$$

1.4. Perturbation theory

Let the weak perturbation yielding the additional term V_{12} in the single-particle Hamiltonian ϵ_{12} be enforced on the system.

Let designations for corrections of first order be introduced

$$\rho_{12}^{(\pm)} = \tfrac{1}{2}\left(\langle a_2^+ a_1 \rangle^{(1)} \pm \langle a_{\tilde{1}}^{\mp} a_{\tilde{2}} \rangle^{(1)}\right) = \rho_{21}^{(\pm)*} = \pm \rho_{\tilde{2}\tilde{1}}^{(\pm)}$$
$$h_{12}^{(\pm)} = \tfrac{1}{2}\left(\langle a_{\tilde{2}} a_1 \rangle^{(1)} \pm \langle a_2^+ a_{\tilde{1}}^{\mp} \rangle^{(1)}\right) = -h_{\tilde{2}\tilde{1}}^{(\pm)} = \pm h_{21}^{(\pm)*} \tag{1.20}$$

and respectively

$$\epsilon_{12}^{(\pm)} = U_{12}^{(\pm)} + V_{12}^{(\pm)} = \pm \sum_{1'2'} \overline{\langle 1\tilde{1}' | G | \tilde{2}'2 \rangle} \rho_{1'2'}^{(\pm)} + \tfrac{1}{2}(V_{12} \pm V_{\tilde{2}\tilde{1}}) = \epsilon_{21}^{(\pm)*} = \pm \epsilon_{\tilde{2}\tilde{1}}^{(\pm)} \tag{1.21}$$
$$\Delta_{12}^{(\pm)} = -\sum \langle 1\tilde{2} | G | \tilde{2}' 1' \rangle h_{1'2'}^{(\pm)} = -\Delta_{\tilde{2}\tilde{1}}^{(\pm)} = \pm \Delta_{21}^{(\pm)*}.$$

Equations (1.11) for the first-order terms in these notations take the form

$$\begin{aligned} &[(\epsilon_2/E_2) + (\epsilon_1/E_1)]\, \rho_{12}^{(\pm)} - [(\Delta_2/E_2) \pm (\Delta_1/E_1)]\, h_{12}^{(\pm)} = 0 \\ &[(\Delta_2/E_2) \pm (\Delta_1/E_1)]\, \rho_{12}^{(\pm)} + [(\epsilon_2/E_2) - (\epsilon_1/E_1)]\, h_{12}^{(\pm)} = 0, \end{aligned} \tag{1.22}$$

the compatibility of which is ensured by equality (1.19'). Equations (1.22) yield the simple relation between $\rho^{(\pm)}$ and $h^{(\pm)}$. It is convenient, therefore, to introduce the new variables, satisfying (1.22) identically.

Let us introduce the values

$$\xi_{12}^{(\pm)} = u_1 u_2 \mp v_1 v_2 \ ; \ \eta_{12}^{(\pm)} = u_1 v_2 \pm v_1 u_2 \tag{1.23}$$

where u, v are defined by the relations

$$u_1^2 - v_1^2 = \epsilon_1/E_1 \ ; \ 2u_1 v_1 = \Delta_1/E_1 \tag{1.23'}$$
$$u_{\tilde{1}} = u_1 \ ; \ v_{\tilde{1}} = -v_1.$$

* Strictly speaking, Δ_1 is a matrix in spin space of the form

$$\Delta_1(\sigma\sigma') = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}. \tag{1.19a}$$

Taking into account that the spin changes its direction under time reversal, i.e.

$$\varphi_{\tilde{1}}(\vec{r}, \sigma) = \varphi_1^*(\vec{r}, -\sigma) \tag{1.19b}$$

it is possible to eliminate the spin coordinate $\sigma = \pm 1$, replacing (1.19b) by

$$\varphi_{\tilde{1}}(\vec{r}) = \gamma_1 \varphi_1^*(\vec{r}) \tag{1.19c}$$

where, for example, $\gamma_1 = 1$ but $\gamma_{\tilde{1}} = -1$. As the non-zero components of $\Delta_1(\sigma\sigma')$ also pass into each other under time reversal we may use (1.19 ···) instead of matrix representation (1.19a).

Then it is clear * that because of (1.22) the quantities $\rho^{(\pm)}$ and $h^{(\pm)}$ can be expressed in the form

$$\rho_{12}^{(\pm)} = \eta_{12}^{(\pm)} \; Z_{12}^{(\pm)}$$
$$h_{12}^{(\pm)} = \xi_{12}^{(\pm)} \; Z_{12}^{(\pm)}. \qquad (1.24)$$

The values $Z_{12}^{(\pm)}$ are reasonably chosen as new unknown quantities. It should be noted that the symmetry properties resulting from (1.20) hold for $Z^{(\pm)}$

$$Z_{12}^{(\pm)} = - Z_{\bar{2}\bar{1}}^{(\pm)} = \pm Z_{21}^{(\pm)*}. \qquad (1.24')$$

Separating terms of first order in equation (1.15) and making simple transformations we obtain

$$(A_{\bar{2}1} \pm A_{12}^{*})^{(1)} / 2\xi_{12}^{(\mp)} = E_{12} \; Z_{12}^{(\pm)} - \xi_{12}^{(\pm)} \; \Delta_{12}^{(\pm)} + \eta_{12}^{(\pm)} \; \epsilon_{12}^{(\pm)} = 0 \qquad (1.25)$$

$$E_{12} = E_1 + E_2.$$

Equation (1.25) defines corrections of the first order for averages from pairs of operators (1.20).

Non-diagonal corrections of first order enter the expression for the ground-state energy in quadratic combinations. Therefore, diagonal corrections to averages $\langle a^{+}a \rangle$ and $\langle aa \rangle$ of the second order should be considered simultaneously. Equations (1.11) have a kinematic nature, as they do not contain the intrinsic characteristic of the system (interaction). Using (1.11) one may establish some general relation between first-and second-order approximations in (1.11). Let us separate second-order terms in (1.11)

$$(1 - 2\langle a_1^{+}a_1 \rangle^{0})\langle a_1^{+}a_1 \rangle^{(2)} - \langle a_1^{+}a_{\bar{1}}^{+} \rangle^{0}(\langle a_{\bar{1}}a_1 \rangle^{(2)} + \langle a_1^{+}a_{\bar{1}}^{+} \rangle^{(2)})$$

$$= \sum_2 (\langle a_1^{+}a_2 \rangle^{(1)}\langle a_2^{+}a_1 \rangle^{(1)} + \langle a_1^{+}a_{\bar{2}}^{+} \rangle^{(1)}\langle a_{\bar{2}}a_1 \rangle^{(1)}) \qquad (1.26)$$

$$\langle a_1^{+}a_{\bar{1}}^{+} \rangle^{0}(\langle a_1^{+}a_1 \rangle^{(2)} - \langle a_{\bar{1}}^{+}a_{\bar{1}} \rangle^{(2)}) = \sum_2 (\langle a_1^{+}a_{\bar{2}} \rangle^{(1)}\langle a_1^{+}a_{\bar{2}}^{+} \rangle^{(1)} - \langle a_1^{+}a_2 \rangle^{(1)}\langle a_2^{+}a_{\bar{1}}^{+} \rangle^{(1)}).$$

Substituting in (1.26) the values for averages of zero and first order and expressing the latter through $Z^{(\pm)}$ according to (1.20) and (1.24), we finally obtain the desired relations

$$W_1 \equiv \tfrac{1}{2}\epsilon_1(\langle a_1^{+}a_1 \rangle^{(2)} + \langle a_{\bar{1}}^{+}a_{\bar{1}} \rangle^{(2)}) - \tfrac{1}{2}\Delta_1(\langle a_{\bar{1}}a_1 \rangle^{(2)} + \langle a_1^{+}a_{\bar{1}}^{+} \rangle^{(2)}) \qquad (1.27)$$

$$= E_1 \sum_2 (|Z_{12}^{(+)}|^2 + |Z_{12}^{(-)}|^2)$$
$$\tfrac{1}{2}(\langle a_1^{+}a_1 \rangle^{(2)} - \langle a_{\bar{1}}^{+}a_{\bar{1}} \rangle^{(2)}) = \sum_2 (Z_{12}^{(-)} Z_{12}^{(+)*} + Z_{12}^{(-)*} Z_{12}^{(+)}). \qquad (1.28)$$

* Relations useful in the calculations should be noted:

$$\eta_{12}^{(\pm)2} = 1 - \xi_{12}^{(\pm)2} = \tfrac{1}{2}[1 - (\epsilon_1\epsilon_2/E_1E_2) \pm (\Delta_1\Delta_2/E_1E_2)]; \eta_{12}^{(\pm)}\eta_{12}^{(\mp)} = \tfrac{1}{2}[(\epsilon_1/E_1) - (\epsilon_2/E_2)]$$

$$\xi_{12}^{(+)}\xi_{12}^{(-)} = \tfrac{1}{2}[(\epsilon_1/E_1) + (\epsilon_2/E_2)]; \xi_{12}^{(\pm)}\eta_{12}^{(\pm)} = \frac{1}{2E_1E_2}(\epsilon_1\Delta_2 \pm \epsilon_2\Delta_1); \xi_{12}^{(+)}\eta_{12}^{(\pm)} = \tfrac{1}{2}[(\Delta_2/E_2) \pm (\Delta_1/E_1)]$$

Let us now find the correction to the ground-state energy. Using decomposition condition (1.9), we have from (1.13)

$$\langle H\rangle = \sum_{12}(\mathcal{E}_{12} - \lambda\delta_{12})\langle a_1^+ a_2\rangle + \tfrac{1}{2}\sum\langle\overline{1\,\tilde{1}'|G|\tilde{2}'2}\rangle\langle a_1^+ a_2\rangle\langle a_{\tilde{1}}^+ a_{\tilde{2}'}\rangle + \tfrac{1}{2}\sum\langle 1\,\tilde{2}|G|\tilde{2}'1'\rangle\langle a_1^+ a_{\tilde{2}}^+\rangle\langle a_{\tilde{2}'} a_{1'}\rangle \tag{1.29}$$

The first-order correction to $\langle H\rangle$ is lacking, as zero approximation is defined from the extremum of $\langle H\rangle$. (It can easily be verified using (1.22)). Second-order terms in (1.29) containing products of $\langle\ \rangle^{(1)}\langle\ \rangle^{(1)}$ can easily be reduced to the form

$$\tfrac{1}{2}\sum\langle\overline{1\,\tilde{1}'|G|\tilde{2}'2}\rangle(\rho_{21}^{(+)}\rho_{1'2'}^{(+)} - \rho_{21}^{(-)}\rho_{1'2'}^{(-)}) + \tfrac{1}{2}\sum\langle 1\,\tilde{2}\,|G|\,\tilde{2}'1'\rangle(h_{21}^{(+)}h_{1'2'}^{(+)} - h_{21}^{(-)}h_{1'2'}^{(-)}). \tag{1.30}$$

Second-order terms with diagonal averages in (1.29) may be reduced to $\sum_1 W_1$, where W_1 is defined in (1.27). Expressing (1.30) by means of (1.25) through $Z^{(\pm)}$ and making use of (1.27) we get

$$\langle H\rangle^{(2)} = \tfrac{1}{2}\sum E_{12}(|Z_{12}^{(+)}|^2 + |Z_{12}^{(-)}|^2) + \tfrac{1}{2}\sum\langle\overline{1\,\tilde{1}'|G|\tilde{2}'2}\rangle(\eta_{12}^{(+)}\eta_{1'2'}^{(+)}Z_{21}^{(+)}Z_{1'2}^{(+)} - \eta_{21}^{(-)}\eta_{1'2}^{(-)}Z_{21}^{(-)}Z_{1'2'}^{(-)}) + \tfrac{1}{2}\sum\langle 1\tilde{2}|G|\tilde{2}'1'\rangle(\xi_{21}^{(+)}\xi_{1'2'}^{(+)}Z_{21}^{(+)}Z_{1'2}^{(+)} - \xi_{21}^{(-)}\xi_{1'2}^{(-)}Z_{21}^{(-)}Z_{1'2'}^{(-)}). \tag{1.30'}$$

Making use of the definition for $\epsilon^{(\pm)}$ and $\Delta^{(\pm)}$ (1.21) we rewrite (1.30') as follows

$$\langle H\rangle^{(2)} = \tfrac{1}{2}\sum_{12} Z_{21}^{(+)}\{E_{12}Z_{12}^{(+)} + \eta_{12}^{(+)}(\epsilon_{12}^{(+)} - V_{12}^{(+)}) - \xi_{12}^{(+)}\Delta_{12}^{(+)}\} - \tfrac{1}{2}\sum_{12} Z_{21}^{(-)}\{E_{12}Z_{12}^{(-)} + \eta_{12}^{(-)}(\epsilon_{12}^{(-)} - V_{12}^{(-)}) - \xi_{12}^{(-)}\Delta_{12}^{(-)}\},$$

from which in view of the equations for $Z^{(\pm)}$ (1.25) we finally get

$$\langle H\rangle^{(2)} = -\tfrac{1}{2}\sum\eta_{12}^{(+)}V_{12}^{(+)}Z_{21}^{(+)} + \tfrac{1}{2}\sum\eta_{12}^{(-)}V_{12}^{(-)}Z_{21}^{(-)}. \tag{1.31}$$

Expression (1.31) defines the correction to the average value of the basic Hamiltonian H. For the perturbating Hamiltonian we obtain in a similar approximation

$$\langle V\rangle^{(2)} = \sum_{12} V_{12}\langle a_1^+ a_2\rangle^{(1)} \tag{1.32}$$

which, after simple calculation, takes the form of

$$\langle V\rangle^{(2)} = \sum_{12}\eta_{12}^{(+)}V_{12}^{(+)}Z_{12}^{(+)} - \sum_{12}\eta_{12}^{(-)}V_{12}^{(-)}Z_{21}^{(-)}. \tag{1.32'}$$

From (1.32') and (1.31) it is seen that

$$\langle H\rangle^{(2)} = -\tfrac{1}{2}\langle V\rangle^{(2)}.$$

1.5. Time-dependent self-consistent field

Hitherto we have been considering equations defining the time-independent ground state of the system. In describing collective excitations it will be useful to introduce the concept of a time-dependent self-consistent field. By collective excitations existing in the system is meant the possibility of separating a special degree of freedom connected with the self-consistent field whose excitation does not interfere with individual particle degrees of freedom. In other words, such quantities as single-particle states, as well as quasi-particles and their occupation numbers ("vacuum" for the ground state), retain their meaning for the excited states of the self-consistent field. In fact, the classification of excitations into collective and single-particle appears to be possible quite approximately, and their interference determines the process of relaxation and damping of collective excitations. In the time-dependent self-consistent field method, however, these effects are neglected.

Instead of (1.12) in the case of a time-dependent self-consistent field the main equations take the form of ($\hbar = 1$)

$$\begin{aligned} i\frac{\partial}{\partial t}\langle a_1\, a_2\rangle &= \langle [a_1\, a_2;\, H]\rangle \equiv A_{12} \\ i\frac{\partial}{\partial t}\langle a_1^+ a_2\rangle &= \langle [a_1^+ a_2;\, H]\rangle \equiv B_{12}. \end{aligned} \tag{1.33}$$

If a time-dependent part of the self consistent field may be considered as perturbation, then equations (1.33) are linearized. In view of (1.20) and (1.24) we obtain in this case

$$i\frac{\partial}{\partial t} Z_{12}^{(\mp)} = E_{12}\, Z_{12}^{(\pm)} - \xi_{12}^{(\pm)} \Delta_{12}^{(\pm)} + \eta_{12}^{(\pm)} \epsilon_{12}^{(\pm)} \tag{1.34}$$

from which for Fourier components $Z_{12}^{(\pm)}(\omega)$ we find

$$\omega\, Z_{12}^{(\mp)}(\omega) = E_{12} Z_{12}^{(\pm)}(\omega) - \xi_{12}^{(\pm)} \Delta_{12}^{(\pm)}(\omega) + \eta_{12}^{(\pm)} \epsilon_{12}^{(\pm)}(\omega). \tag{1.35}$$

Here, according to (1.20) and (1.21),

$$\begin{aligned} Z_{12}^{(\pm)}(\omega) &= -\, Z_{\tilde{2}\tilde{1}}^{(\pm)}(\omega) = \pm\, Z_{21}^{(\pm)*}(-\omega) \\ \Delta_{12}^{(\pm)}(\omega) &= -\,\Sigma \langle 1\tilde{2} | G | \tilde{2}'1' \rangle \xi_{1'2'}^{(\pm)} Z_{1'2'}^{(\pm)}(\omega) = -\,\Delta_{\tilde{2}\tilde{1}}^{(\pm)}(\omega) = \pm\, \Delta_{21}^{(\pm)*}(-\omega) \\ \epsilon_{12}^{(\pm)}(\omega) &= \pm\, \Sigma \overline{\langle 1\tilde{1}' | G | \tilde{2}'\tilde{2} \rangle}\, \eta_{1'2'}^{(\pm)} Z_{1'2'}^{(\pm)}(\omega) + V_{12}^{(\pm)}(\omega) = \epsilon_{21}^{(\pm)*}(-\omega) = \pm\, \epsilon_{\tilde{2}\tilde{1}}^{(\pm)}(\omega). \end{aligned} \tag{1.36}$$

It should be noted that equations (1.35) under transformation $1 \rightleftarrows 2$ and $\omega \to -\omega$ become complex conjugated equations. Thus it follows that the dispersion equation for (1.35) defines the frequency ω except for the sign.

After writing the basic equations let us consider certain problems connected with the collective properties of nuclei.

2. THE EQUILIBRIUM SHAPE OF NUCLEI

2.1. Formulation of the problem

Numerous models have been suggested recently to describe the excitations of those nuclei where the spectrum essentially differs from that of ro-

tation (spherical oscillating nucleus [12], non-axial rotator [13], model with γ-unstable potential [14] and others [15]). Although all these models make use of the essentially different concepts concerning the equilibrium shape of the nuclei, their predictions are quite close and experiment does not at present offer any possibility of choice in favour of any definite model. In such a situation the estimation of the equilibrium shape of the nucleus from its intrinsic structure is of special interest.

Deformation in nuclei results from the effect of polarization by nucleons in the upper unfilled shells, hence the problem of the equilibrium shape of the nucleus is reduced mainly to the definition of the equilibrium configuration of these nucleons. If we temporarily neglect the polarization of the closed-shell core and consider external nucleons to be in a spherical potential well, then their configuration is determined by the nature of their interaction. As was stated above, the main role in this case is played by the two parts of interaction: Cooper-pairing and the self-consistent field. When the closed shell is polarized, the external nucleons are affected not only by their own self-consistent field, but also by the additional field of the closed-shell core. But as these fields possess an identical symmetry and are proportional to each other, the closed-shell polarization effect is equivalent to the increase of the eigen-self-consistent field of the external nucleons, and its calculation is reduced to the renormalization of their interaction constant.

In the deformed nucleus the grouping into external nucleons and closed-shell core is rather conditional. Let us assume that the region "close to the Fermi surface" lies in the energy interval $\lesssim \hbar v/R$ (v - velocity at Fermi surface) and contains $\sim A^{2/3}$ one-particle states. In the slightly deformed nucleus $\hbar v/R$ coincides with the intershell distance. In this case, let the external nucleons be particles in the upper unfilled shell. It should be noted that the double inequality $\Delta \ll \hbar v/R \ll \mathcal{E}_F$ is valid. Thus, leaving the question of interaction renormalization aside, we shall consider only $\sim A^{2/3}$ nucleons close to the Fermi surface taking their Cooper pairing and the self-consistent field into consideration.

2.2. Choice of nucleon-nucleon interaction

Let us assume that the spherically symmetrical part of the self-consistent field is already taken into consideration in the original single-particle Hamiltonian ϵ, and the residual interaction results only in the deviation from spherical symmetry. Since the most essential role in nuclei is played by quadrupole anisotropy we may assume the additional self-consistent field to be of quadrupole symmetry, i.e.

$$\sum_{\tilde{1}\tilde{2}} \overline{\langle 1\tilde{1}|G|\tilde{2}2\rangle}\langle a^+_{\tilde{1}} a_{\tilde{2}}\rangle = -\kappa \sum_{\mu} Q^*_{\mu}\langle 1|q_{\mu}|2\rangle, \qquad (2.1)$$

where Q_μ and single-particle operator q_μ possess the property of spherical tensor operators of the second rank. The right-hand side of Eq. (2.1) can be easily interpreted as an interaction between a single-particle moment q_μ and the total moment Q_μ of the other particles. We can imply such an interaction to be a reflection of nucleon-nucleon "quadrupole interaction" of the form

$$G_Q(\vec{r}_1, \vec{r}_2) = -\kappa \sum_{\mu} q^*_{\mu}(\vec{r}_2)\, q_{\mu}(\vec{r}_1). \qquad (2.1')$$

The real interaction entering the left-hand side of (2.1) is not reduced only to (2.1'). In the general case $G(\vec{r}_1, \vec{r}_2)$ can be expressed as a series of scalar products of spherical tensor operators of different ranks. Equation (2.1) implies in fact that density matrix $\langle a_1^+ a_{2'} \rangle$, which possesses quadrupole symmetry, picks out the only quadrupole term from this sum*

In view of interaction (2.1) the one-particle Hamiltonian (1.16) takes the form

$$\epsilon(12) = \mathcal{E}_{12} - \lambda\delta_{12} - \kappa \sum_{\mu} Q_{\mu}^{*} \langle 1 | q_{\mu} | 2 \rangle, \tag{2.2}$$

where the total quadrupole moment of the nucleons Q_μ is defined by

$$Q_{\mu} = \sum_{\nu\nu'} \langle \nu | q_{\mu} | \nu' \rangle \langle a_{\nu}^{+} a_{\nu'} \rangle = (-1)^{\mu} Q_{-\mu}^{*}. \tag{2.3}$$

Let us choose as basic function of secondary quantization eigenfunctions $|\nu\rangle$ of Hamiltonian (2.2) at the fixed value of parameter Q_μ

$$\hat{\epsilon} |\nu\rangle \equiv (\hat{\mathcal{E}} - \lambda - \kappa \sum_{\mu} Q_{\mu}^{*} \hat{q}_{\mu}) |\nu\rangle = \epsilon_{\nu}(Q) |\nu\rangle. \tag{2.4}$$

The first of conditions (1.18) is now fulfilled. Let us assume that the second condition is also valid and put (see (1.19'''))

$$\Delta(12) = \gamma_1 \Delta \delta_{2\tilde{1}}, \tag{2.5}$$

where Δ is constant. The interaction yielding the main contribution to Cooper pairing and resulting in (2.5) may be represented as

$$\langle 1\tilde{2} | G | \tilde{2}'1' \rangle = - G \gamma_1 \gamma_{1'} \delta_{12} \delta_{1'2'}. \tag{2.6}$$

A set of matrix elements (2.6) is usually called pairing interaction.**

2.3. Equations defining the equilibrium deformation

In fulfilling conditions (2.4) and (2.5), equations (1.19) define average values of pairs of operators by the three parameters: Δ, chemical potential λ and quadrupole moment Q_μ. The equation for λ is found from the expression for the number of particles N:

$$N = \mathrm{Sp}(\hat{\rho}) = \tfrac{1}{2} \sum_{\nu} (1 - \epsilon_{\nu}/E_{\nu}). \tag{2.7}$$

For the definition of Δ and Q_μ we get from (1.19), (2.3) and (2.5)

$$G \sum_{\nu} (1/2E_{\nu}) = 1 \tag{2.8}$$

$$Q_{\mu} = \mathrm{Sp}(\hat{\rho}\hat{q}_{\mu}) = \tfrac{1}{2} \sum_{\nu} \langle \nu | q_{\mu} | \nu \rangle (1 - \epsilon_{\nu}/E_{\nu}). \tag{2.9}$$

* Strictly speaking, the quadrupole term from the expansion of $G(\vec{r}_1, \vec{r}_2)$ in spherical tensor operators has the form of (2.1') but with κ depending on $|\vec{r}_1|$, $|\vec{r}_2|$. We shall neglect the inessential dependence and take κ as constant. We may also include the radial factor r^2 into q_μ and identify it with the single-particle quadrupole moment (redefining κ).

** Parameter G must be reasonably considered as phenomenological and taken from experiments. In this case its values for the deformed and spherical nuclei (in the same A-region) appear to be very close. Therefore, one may consider G independent of Q.

Knowing the averages of pairs of operators, the ground-state energy can easily be found. As a result of using the above formulae we obtain

$$W = \langle H \rangle + \lambda N = \sum_{\nu} \rho_{\nu\nu} (\epsilon_{\nu} + \lambda) + \tfrac{1}{2} \kappa \sum_{\mu} Q_{\mu}^{*} Q_{\mu} - \Delta^2/2G. \qquad (2.10)$$

We shall make an important remark. If Eq. (2.9) is temporarily not taken into consideration, parameters Q_{μ} remain free. In this case equations (2.4), (2.7) and (2.8) coincide with analogous equations in the problem treating nucleons in the external potential well with deformation parameters Q_{μ}. In such a problem one may consider the ground-state energy $W(Q_{\mu})$ and investigate its dependence on Q_{μ}. If expression (2.10) is used for $W(Q_{\mu})$ it can easily be seen that the positions of the extrema will be defined by equation (2.9). Indeed, differentiating (2.10) over Q_{μ} and assuming λ and Δ to be functions of Q_{μ} defined by equations (2.8) and (2.9), we get

$$(\partial W / \partial Q_{\mu}^{*}) = \kappa (Q_{\mu} - \mathrm{Sp}(\hat{\rho}\, \hat{q}_{\mu})); \qquad (2.11)$$

therefore, equation $\partial W / \partial Q_{\mu}^{*} = 0$ coincides with (2.9). The latter, in this case, may be treated as condition of consistency: only such deformations are physically allowable which coincide with the density deformation of the nucleons. It should be stressed that the function $W(Q)$, strictly speaking, has physical meaning only in the extremum points. In particular, the second derivatives from $W(Q_{\mu})$

$$(\partial^2 W / \partial Q_{\mu}^{*} \partial Q_{\mu'}) = \kappa (\delta_{\mu\mu'} - \frac{\partial}{\partial Q_{\mu'}} \mathrm{Sp}\,(\hat{\rho}\, \hat{q}_{\mu})) \equiv \kappa\, (1-R)_{\mu\mu'} \qquad (2.12)$$

need not be identified with the physical tensor of the quadrupole restoring force; the latter may be calculated from the variation of energy of the system in the external field.* As a result, we find

$$K_{\mu\mu'} = \kappa \{R^{-1}(1-R)\}_{\mu\mu'} \qquad (2.13)$$

which differs from (2.12). In order to investigate the stability of extremum states it is quite sufficient to determine the sign of K only, so the difference between (2.12) and (2.13) is not essential for us.

2.4. Some general conclusions

Let us proceed to the investigation of the equation for Q_{μ}. First of all, it should be noted that Eq. (2.9) defines components Q_{μ} only with an accuracy up to the random orientation of axis. The nuclear shape is characterized by the two invariants

$$\sum_{\mu} Q_{\mu}^{*} Q_{\mu} = \beta^2 \qquad (2.14)$$

$$\beta^3 - (\sqrt{7}/2)^{1/2} \sum_{(\mu)} (2\mu'\, 2\mu'' | 2\mu)\, Q_{\mu}\, Q_{\mu'}^{*} Q_{\mu''}^{*} = \beta^3 \cos 3\gamma,$$

where $(2\mu'\, 2\mu'' | 2\mu)$ are Clebsch-Gordan coefficients. In analogy with (2.14) let us introduce scalar products of the operator of the single-particle quadrupole moment $\hat{q}_{\mu}$ and the combination Q_{μ}^{*}/β

* In [5] the restoring force was estimated by the Lagrange multiplier method, which yields a similar result.

$$\hat{q} = \sum_{\mu} \hat{q}_{\mu} Q_{\mu}^{*} / \beta \qquad (2.14')$$

$$\hat{r} = -\left(\frac{7}{2}\right)^{1/2} \sum_{(\mu)} \hat{q}_{\mu} \, (2\mu' 2\mu'' | 2\mu) \, Q_{\mu'}^{*} Q_{\mu''}^{*} / \beta^{2} .$$

Then from (2.9) we obtain for parameters β and γ the following equations:

$$\beta = \mathrm{Sp}\,(\hat{\rho}\,\hat{q}) \qquad (2.15)$$

$$\beta \cos 3\gamma = \mathrm{Sp}\,(\hat{\rho}\,\hat{r}). \qquad (2.15')$$

If one chooses for coordinate axis the main axis of the ellipsoid Q_{μ} ($Q_0 = \beta \cos\gamma$; $Q_1 = Q_{-1} = 0$; $Q_2 = Q_{-2} = (1/\sqrt{2})\,\beta \sin\gamma$), then

$$\langle \nu | \hat{q} | \nu \rangle = \langle \nu | q_0 | \nu \rangle \cos\gamma + \sqrt{2} \langle \nu | q_2 | \nu \rangle \sin\gamma \qquad (2.16)$$

$$\langle \nu | \hat{r} | \nu \rangle = \langle \nu | q_0 | \nu \rangle \cos 2\gamma - \sqrt{2} \langle \nu | q_2 | \nu \rangle \sin 2\gamma .$$

One may draw several conclusions from the form of equations (2.9) and (2.15). First, there always exists the solution corresponding to sphere ($\beta = 0$). In fact, in spherical nucleus density matrix ρ is independent of the projection of single-particle angular momentum, and $\sum_{m} \langle \nu | q_{\mu} | \nu \rangle = 0$, therefore the right sides of equations (2.15) or (2.9) when $\beta = 0$ ($Q_{\mu} = 0$) become zero. It should also be noted that if the first equation of (2.16) has a solution for β when $\gamma = 0$ or $\gamma = \pi/3$ then the second equation is also satisfied since when $\gamma = 0, \pi/3$ both equations (2.15) coincide (this is seen from (2.16)). This means that there may exist solutions with $\gamma = 0$ or $\pi/3$ corresponding to axially symmetric deformation of the nucleus.

2.5. The region of small deformations

The right sides of equation (2.15) are the quadrupole moments of nuclear density in the external well with deformation (β, γ). In order to determine the nature of the dependence of $\mathrm{Sp}(\hat{\rho}\hat{q})$ on β and γ it is useful to consider the limiting regions of large and small deformations. The ratio of shell level splitting $\kappa Q \langle \nu | q | \nu \rangle \sim \kappa \beta q$ and pairing energy Δ may serve as a useful criterion for the deformation value. The deformation parameter usually defined, $\delta R/R$, is related to β by the equation $\kappa \beta q \simeq \epsilon_F \delta R/R$, where ϵ_F - is the Fermi energy and q is an average absolute value of $\langle \nu | q | \nu \rangle$. By large and small deformations we mean the cases $\delta R/R \ll \Delta/\epsilon_F$ and $\delta R/R \gg \Delta/\epsilon_F$ respectively. (Let us recall that for the real strongly deformed nuclei $\delta R/R \approx A^{-1/3}$, i.e. the absolute deformation value $\delta R/R$ is always small). Also, noting that $\beta \sim A^{2/3}$ when $\delta R/R \approx A^{-1/3}$ we get the estimate $\kappa \simeq \epsilon_F / q^2 A$.

In the region of small deformations the expansion of the right side of (2.9) commences with the linear terms Q_{μ} and in view of the invariance has the following form

$$\mathrm{Sp}(\hat{\rho}\,\hat{q}_{\mu}) = a\, Q_{\mu} - \tfrac{1}{2} b (7/2)^{\frac{1}{2}} \sum_{\mu'\mu''} (2\mu' 2\mu'' | 2\mu)\, Q_{\mu'} Q_{\mu''} + \ldots . \qquad (2.17)$$

where a and b are coefficients independent of Q_{μ}. Hence, for the right side of equation (2.15) we obtain

$$\mathrm{Sp}(\hat{\rho}\hat{q}) = a\,\beta + \frac{1}{2} b\,\beta^{2} \cos 3\gamma + \ldots . \qquad (2.17')$$

and for a and b we have

$$a = \left[\frac{\partial}{\partial \beta} \mathrm{Sp}(\hat{\rho}\hat{q}) \right]_{\beta=0}; \quad b = \left[1/\cos 3\gamma \right] \left[\frac{\partial^2}{\partial \beta^2} \mathrm{Sp}(\hat{\rho}\hat{q}) \right]_{\beta=0}. \qquad (2.17'')$$

The calculation of (2.17'') is simplified if we take into consideration that according to (2.4) and (2.14')

$$\partial \epsilon_\nu / \partial \beta = \langle \nu | \partial \hat{\epsilon} / \partial \beta | \nu \rangle = -\kappa \langle \nu | \hat{q} | \nu \rangle - \partial \lambda / \partial \beta \qquad (2.18)$$

and derivatives $\partial \Delta / \partial \beta$ and $\partial \lambda / \partial \beta$ are zero when $\beta = 0$. As a result of simple calculations we get

$$a = a_1 + a_2 = \kappa \sum_\nu (\Delta^2 / 2E_\nu^3)\, q^2_{\nu\nu} + \sum_\nu \rho_{\nu\nu} (\partial q_{\nu\nu} / \partial \beta) \qquad (2.19)$$

$$b \cos 3\gamma = \frac{3}{2} \kappa^2 \sum_\nu (\Delta^2 \epsilon_\nu / E_\nu^5)\, q^3_{\nu\nu} + \frac{3}{2} \kappa \sum_\nu (\Delta^2 / E_\nu^3) q_{\nu\nu} (\partial q_{\nu\nu} / \partial \beta) \qquad (2.19')$$

$$+ \sum_\nu \rho_{\nu\nu} (\partial^2 q_{\nu\nu} / \partial \beta^2)$$

where the expressions in the right sides are taken when $\beta = 0$. For derivatives from $q_{\nu\nu} = \langle \nu | \hat{q} | \nu \rangle$ entering (2.19) the exact formulae is easily obtained (differentiating equation (2.4)):

$$\partial q_{\nu\nu} / \partial \beta = 2 \kappa \sum_{\nu'}{}' [\, |q_{\nu\nu'}|^2 / (\epsilon_{\nu'} - \epsilon_\nu)\,]. \qquad (2.20)$$

Let us estimate both terms in (2.19). The value $\Delta^2 / 2E_\nu^3$ notably differs from zero in the region $\Delta \sim \mathcal{E}_F A^{-2/3}$ wide inside which lie $\sim A^{+1/3}$ single-particle states. This yields the estimate for the first term in (2.19), $a_1 \sim (\kappa q^2 / \mathcal{E}_F) A \sim 1$. On the other hand from (2.20) follows $\partial q_{\nu\nu'} / \partial \beta \sim \kappa q^2 / \mathcal{E}_F$, so the second term in (2.19) in the case of summation over $A^{2/3}$ states at the Fermi surface can be estimated as $a_2 \sim a_1 A^{-1/3} \ll a_1$. Analogous estimates show that the two latter terms in (2.19'') are a factor $A^{-1/3}$ smaller than the former one. Thus, for particles at the Fermi surface one may conclude*

$$a \approx \kappa \sum_\nu (\Delta^2 / 2E_\nu^3) q^2_{\nu\nu}; \; b \cos 3\gamma \approx \frac{3}{2} \kappa^2 \sum_\nu (\Delta^2 \epsilon_\nu / E_\nu^5)\, q^3_{\nu\nu}. \qquad (2.21)$$

Equations (2.15), (2.17), beside the spherical solution $\beta = \beta_0 = 0$, also possess the formal solution

$$\beta = \beta_1 = 2(1-a)/b \cos 3\gamma \qquad (2.22)$$

which, with a closely approaching unity, lies in the region of small β considered. To determine the stability of these solutions we shall find coefficients of the restoring force with respect to the change of variables β and γ. From (2.13) and (2.17) for small Q_μ ($|1-a| \ll 1$) it would be easy to obtain

$$K_{\mu\mu'} = \kappa (1-a)\, \delta_{\mu\mu'} + \kappa b (7/2)^{\frac{1}{2}} \sum_{\mu_1} (2_\mu\, 2_{\mu_1} | 2_{\mu'}) Q_{\mu_1}. \qquad (2.23)$$

---

* If summing is made in a_2 over all states, then a_2 and a_1 will be of the same order. Term a_2 in this case results in the renormalization of quadrupole interaction constant κ.

The transition to variables β and γ is carried out by means of formulae (following from (2.14))

$$\partial Q_\mu/\partial\beta = Q_\mu/\beta;\ \partial Q_\mu/\partial\gamma = Q_\mu \operatorname{ctg} 3\gamma + (7/2)^{\frac{1}{2}}(1/\beta \sin 3\gamma) \sum_{\mu'\mu''} (2_{\mu'} 2_{\mu''} | 2_\mu) Q_{\mu'} Q_{\mu''} \tag{2.24}$$

Substituting (2.23) and (2.24) in equality

$$K_{xy} = \sum_{\mu\mu'} K_{\mu\mu'} (\partial Q_\mu/\partial x)(\partial Q^*_{\mu'}/\partial y);\ (x, y = \beta, \gamma), \tag{2.24'}$$

we obtain

$$K_{\beta\beta} = \kappa(1-a) - \kappa b \beta \cos 3\gamma$$

$$(1/\beta^2)K_{\gamma\gamma} = \kappa(1-a) + \kappa b \beta \cos 3\gamma. \tag{2.25}$$

For the spherical state ($\beta = 0$) the restoring force regarding variables β and $\beta\gamma$ is equal, as one may expect from the isotropy, and the stability condition takes the form of $a < 1$. As is clear from (2.21), the a value is widely dependent on the upper shell filling. It reaches maximum near the middle of the shell ("maximum filling") and is zero for the closed shell (where $\Delta = 0$). Thus, with a small number of particles (or holes) in the upper unfilled shell the spherical state is stable up to a critical filling for which $a = a_c = 1$. With a further increase of the filling the spherical nucleus becomes unstable.

The coefficients of elasticity in the stationary point (2.22) are equal to

$$K_{\beta\beta}(\beta_1) = \kappa(a-1)$$

$$K_{\gamma\gamma}(\beta_1) = -3\kappa(a-1)\beta_1^2 \tag{2.26}$$

from which it is clear that with sign $(a-1)$ either $K_{\beta\beta}$ or $K_{\gamma\gamma}$ is negative. Thus, the state with $\beta = \beta_1$ (2.22) is always unstable.

2.6. The region of large deformations ($\delta R/R \gg \Delta/\mathcal{E}_F$)

It should be noted that calculation of $\mathrm{Sp}(\hat\rho \hat q_\mu)$ in this limiting case is equivalent to the problem of the quadrupole moment of non-interacting particles ($\Delta \to 0$) in a weakly deformed well ($\delta R/R \ll 1$). In the quasi-classical picture, when the distribution of particles in coordinate space is described by the average density, defined by the shape of the potential well only, the $\mathrm{Sp}(\hat\rho \hat q_\mu)$ value is proportional to Q_μ and does not depend on the characteristic of the filled single-particle levels. Density fluctuations, connected with particles at the Fermi surface and, consequently, dependent on their quantum numbers, result in the additional contribution to the quadrupole moment. This part $\mathrm{Sp}(\hat\rho \hat q_\mu)$ is very sensitive to the sequence of levels at the Fermi surface, which is mainly defined by the shape of the deformation, parameter γ, and much less by the value of deformation β. Thus, a change of γ from 0 to $\pi/3$ inverts the order of levels, while small variations of β, in fact, do not change it. As a result of the above, the right side of (2.15) in the region of large deformations may be shown as follows

$$\mathrm{Sp}(\hat\rho \hat q_\mu) \approx B(\gamma) + C\beta. \tag{2.27}$$

Here the term $B(\gamma)$ is related to the particle density fluctuations at the Fermi surface. It is sensitive to the shell occupation and, therefore, unmonotonously dependant on the number of nucleons A. The last term in (2.27) is of a quasi-classic nature, so one may assert that the coefficient C is smoothly and monotonously dependant on A.

Expression (2.27) may be approached in another way. One may easily be convinced that the tensor $\mathrm{Sp}(\hat{\rho}\hat{q}_\mu)$ is the derivative over $-\kappa Q_\mu^*$ from the scalar function

$$U = \sum_\nu \rho_{\nu\nu} (\epsilon_\nu + \lambda) - \Delta^2 / 2G. \qquad (2.28)$$

(λ and Δ are functions of Q_μ, defined from Eqs (2.7) and (2.8)). The value of (2.28) is the nucleon energy in the external well in view of their pairing interaction. This energy depends only on invariant quantities β and γ, containing under large deformations the linear and quadratic in β terms

$$-U \approx B(\gamma)\beta + \tfrac{1}{2} C\beta^2 \qquad (2.29)$$

where the first term is the nucleon energy in the upper unfilled shells and the last term is related mainly to the closed-shell particles. The derivative from (2.29) in β yields (2.27).

From (2.27) one may see that the value C is equal to $(\partial/\partial\beta)\,\mathrm{Sp}(\hat{\rho}\hat{q})$, i.e. is analogous to (2.17'') in the region of large deformations. Simple analysis shows that in differentiating $\mathrm{Sp}(\hat{\rho}\hat{q})$ over β, the quantities λ and Δ may be considered to be constant* so that the expression for C formally coincides with (2.19). In view of (2.20) we obtain

$$C = (\partial/\partial\beta)\,\mathrm{Sp}(\hat{\rho}\hat{q}) = \kappa \sum_\nu (\Delta^2/2E_\nu^3)\, q_{\nu\nu}^2 + \kappa \sum_{\nu\nu'}{}' [(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu)] \left| q_{\nu\nu'} \right|^2. \qquad (2.30)$$

It should be noted that $(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu) \to \Delta^2/2E_\nu^3$ when $\epsilon_\nu \to \epsilon_{\nu'}$, so the first term in (2.30) is the sum of diagonal terms, omitted in the second sum. Joining both terms we obtain**

$$C = (\partial/\partial\beta)\,\mathrm{Sp}(\hat{\rho}\hat{q}) = \kappa \sum_{\nu\nu'}{}' [(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu)] \left| q_{\nu\nu} \right|^2. \qquad (2.31)$$

The quantity $(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu)$ as a function of ϵ_ν with $(\epsilon_{\nu'} - \epsilon_\nu)$ fixed has a sharp maximum at the Fermi surface ($\epsilon = 0$), the width of which is determined by the larger of the values $(\Delta, |\epsilon_{\nu'} - \epsilon_\nu|)$. The area under the curve is equal to unity. We can consider the matrix element $q_{\nu\nu'}$ to be a smoothly varying function of ϵ_ν for fixed values of $(\epsilon_{\nu'} - \epsilon_\nu)$. This makes it possible to perform the substitution

$$(\rho_\nu - \rho_{\nu'})/(\epsilon_{\nu'} - \epsilon_\nu) \to \delta(\epsilon_\nu)$$

---

* This is because the variations of Δ and λ under deformation are small compared with the shift of single-particle levels $\partial\epsilon/\partial\beta \sim \kappa q$. Evaluations show that $\partial\lambda/\partial\beta < \kappa q\, A^{-1/3}$ and $\partial\Delta/\partial\beta \sim \kappa q A^{-1}$. An exception is an oscillator model where $\partial\lambda/\partial\beta \sim \partial\epsilon/\partial\beta$. Taking the terms $\partial\lambda/\partial\beta$ into account results in the compensation of the first term in (2.30) for this model.

** Let us be more exact. The unification of both terms is possible if we take all nucleons into consideration. The second term in (2.30) may be neglected for nucleons at the Fermi surface. Doing this, the formulae in (2.34) will yield not the total deformation, but only that part of it which is associated with the external nucleons.

in (2.31) (see analogous calculations in [6]). After this the sum over ν' in (2.31) can be performed and, introducing the density of particles in the energetic and coordinate space (on the Fermi surface)

$$\rho(\mathcal{E}_F, \vec{r}) = \sum_\nu \varphi_\nu^*(\vec{r})\,\delta(\epsilon_\nu)\,\varphi_\nu(\vec{r}),$$

we find

$$C = \kappa \int d\vec{r}\ \hat{q}^2\ \rho(\mathcal{E}_F; \vec{r}) = \kappa\,\rho(\mathcal{E}_F)\overline{q^2}. \qquad (2.32)$$

Here the average from $\hat{q}^2$ with an accuracy up to $\sim \delta R/R$ values does not differ from the average over spherical distribution, so one may assume that $\overline{q^2}$ is independent of the deformation. The value C, as was expected, does not depend on the deformation and is a smooth function of the number of nucleons A. The quantity C may be connected with the quasi-classical coefficient of elasticity. And, indeed, from (2.27) we have (see (2.12)) $R_{\mu\mu'} \approx C\delta_{\mu\mu'}$, from which, in view of (2.13) and (2.24'') we find

$$K_{\beta\beta} \approx \kappa[(1/C)-1]. \qquad (2.33)$$

From (2.33) it follows that for the stable system the inequality $C < 1$ should be fulfilled.

The solution of equation (2.15) in the region of large deformations in view of (2.27) takes the form

$$\beta = \beta_2 = B(\gamma)/(1-C). \qquad (2.34)$$

The quantitative calculation of the equilibrium deformation (2.34) cannot be carried out in the general form, as the function $B(\gamma)$ is sensitive to the model of single-particle levels. We shall discuss formula (2.33) later in connection with the vibrations of deformed nuclei.

2.7. Phase transition from spherical to deformed nuclei

Let us consider the nature of transition from spherical nuclei to the deformed ones qualitatively. It would be convenient for the analysis to write down equation (2.15) as follows

$$1/a = \Lambda(\beta) \qquad (2.35)$$

where the function $\Lambda(\beta)$ in the region of low deformations according to (2.17') is of the form

$$\Lambda(\beta) = 1 + (b/2a)\,\beta\cos 3\gamma + \ldots. \qquad (\beta \ll \Delta/\kappa q) \qquad (2.36)$$

and under large deformations (see (2.27)) is equal to

$$\Lambda(\beta) \approx C\,a^{-1} + B(\gamma)\,a^{-1}\beta^{-1} \qquad (\beta \ll \Delta/\kappa q). \qquad (2.37)$$

As one sees from (2.36) the behaviour of $\Lambda(\beta)$ in the region of small β depends on the sign of the expression $(b/a)\cos 3\gamma$. According to (2.21) the value a

is always positive and b may change its sign with shell filling.* The extreme expressions (2.36) and (2.37) permit qualitative presentation of the function $\Lambda(\beta)$ behaviour, also in the intermediate region.

Figure 1 plots function $\Lambda(\beta)$ for cases of positive (I) and negative (II) values b cos 3γ. Solutions of equations (2.35), representing extremum values β by fixed values of γ (extremum points of function $W(\beta)$ (2.10))are given by the intersection points of the curve $\Lambda = \Lambda(\beta)$ with horizontal straight line $\Lambda = 1/a$. Moreover, one must take into account the solution $\beta = 0$, excluded from (2.35). As was stated the $\beta = 0$ value corresponds to the minimum of $W(\beta)$ when $a < 1$ ($\Lambda > 1$ in Fig. 1) and to maximum when $a > 1$ ($\Lambda < 1$).

From Fig. 1 one can easily see the full picture of the distribution of the extremum points for $W(\beta)$.

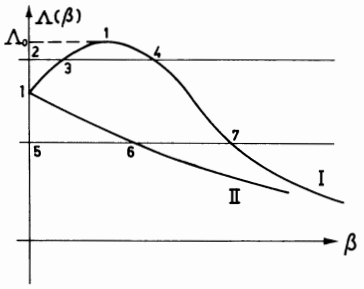

Fig. 1

I. b cos $3\gamma > 0$ (curve I. Fig. 1). In this case one should distinguish three different regions of shell filling (values a):

(1) $1/a > \Lambda_0 = \max \Lambda(\beta)$. One minimum when $\beta = 0$.
(2) $\Lambda_0 > 1/a > 1$. Three extremums: minimum when $\beta = 0$ (2) -maximum (3) -minimum (4).
(3) $1 > 1/a$. Maximum when $\beta = 0$ (5) and minimum for $\beta \neq 0$ (7).

II. b cos $3\gamma < 0$ (curve II, Fig. 1). Two regions of filling

(1) $1/a > 1$. Minimum when $\beta = 0$.
(2) $1 > 1/a$. Maximum when $\beta = 0$ (5) and minimum when $\beta \neq 0$ (6).

Thus, in a region of sufficiently small a in both cases $W(\beta)$ possesses the actual minimum when $\beta = 0$, which corresponds to the stable spherical state. On the other hand, in a region of sufficiently large a there exists only the minimum when $\beta \neq 0$.

Let us consider now the case of transition from spherical nuclei to deformed ones with the increase of filling (decrease $1/a$). When b cos $3\gamma < 0$ transition occurs when $1/a = 1$ and deformation beyond the transition point smoothly rises from zero ("phase transition of second kind"). In the case of b cos $3\gamma > 0$ (curve I) with increase of filling (lowering of horizontal line in Fig. 1) the function $W(\beta)$ acquires a second minimum for $\beta \neq 0$ (4), at first, besides the minimum in the point $\beta = 0$, which then gradually decreases. The transition to the deformed state will occur at the moment when both minima

* For a single j-level b is negative in filling the first half of the shell and is positive for the second half. In the oscillator model we have reversed behaviour. This is associated with the different directions of the level density gradient in these models. Real nuclei in this respect are nearer to the oscillator model.

become equal. In this case deformation proceeds from zero to some finite value ("phase transition of first kind").

Hitherto we have considered the extrema of $W(\beta)$ at fixed γ. Now we shall make use of the stability criterion with respect to the variation of parameter γ. As one sees from (2.26) in the region of small β the sign of the restoring force coefficient $K_{\gamma\gamma}$ is defined by the (1-a) value. Thus solutions for small β corresponding to curve II in Fig. 1, for which $1/a < 1$, appear to be γ-unstable. Therefore the transition to the deformed nucleus may occur only according to curve I. Thus the transition from spherical nuclei to deformed ones occurs with the jump in deformation and is, therefore, a phase transition of the first kind.

3. DO γ-DEFORMED NUCLEI EXIST?

3.1. Calculations in harmonic oscillator model

As stated above, the estimation of the equilibrium shape of the nucleus in the region of large deformations in the general form appears to be impossible.

Let us make use of the oscillator model for the single-particle potential. In this model the matrix elements of the single-particle quadrupole moment q_μ are diagonal inside one shell, and Eq. (2.4) is satisfied for the states $|\nu\rangle = |n_x n_y n_z\rangle$ (n_i - oscillator quantum numbers). State $\tilde{\nu}$ differs only by the spin direction.

For single-particle energies $\epsilon_\nu(\beta, \gamma)$ from (2.4) it follows that

$$\epsilon_\nu = \epsilon - \kappa\beta\,(q_\nu \cos\gamma + s_\nu \sin\gamma) \tag{3.1}$$

where $\epsilon = \hbar\omega n - \lambda$, and matrix elements q_ν and s_ν are equal* to

$$q_\nu = n_z/n - 1/3\,;\; s_\nu = (n_x - n_y)/\sqrt{3}n;\; n = \sum_i n_i. \tag{3.2}$$

Further calculations will be carried out with quasi-classical accuracy neglecting values of order $1/n$. Then summing over $(n_x n_y n_z)$ within the shell $n = \text{const}$ may be replaced by integrating over q_ν and s_ν

$$\sum_\nu \to 2\sqrt{3}\Omega \iint dq\,ds \tag{3.3}$$

where $2\Omega = n^2$ is the number of states in the shell under consideration. The region of integration in plane (q, s) is an equilateral triangle with the vertic in the points $(2/3; 0)$ and $(-1/3; \pm 3^{-\frac{1}{2}})$ (Fig. 2).

Let us first consider equation (2.7) for the chemical potential λ, i.e. for parameter ϵ in (3.1). In the region of large deformations ($\beta \gg \Delta/\kappa$) we may replace the density matrix $\rho_{\nu\nu}$ by the stepping function

$$\rho_{\nu\nu} = \begin{cases} 1 & \epsilon_\nu < 0 \\ 0 & \epsilon_\nu > 0 \end{cases} \tag{3.4}$$

and integrate over the region $\epsilon_\nu(q, s) < 0$. The equation defining the boundary $\epsilon_\nu(q, s) = 0$ determines the straight line which, when $0 < \gamma < \pi/3$, possesses

* We define q_ν, s_ν and consequently Q as dimensionless quantities. Dimension factors are included in parameter κ.

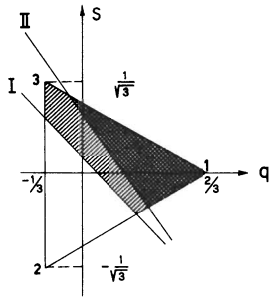

Fig. 2

negative slope to the axis q. When the line passes through vertex 1 the integration region vanishes. This is in accord with the unoccupied shell and corresponds to $\epsilon = \epsilon_{max} = (2/3)\kappa\beta\cos\gamma$. When the shell is fully occupied ($\epsilon_\nu(q, s) = 0$ passes through vertex 2) then $\epsilon = \epsilon_{min} = (2/3)\kappa\beta\cos[\gamma + (2\pi/3)]$. Thus, parameter ϵ varies when the shell is filled in the limits

$$\neq (2/3)\kappa\beta\cos[\gamma + (2\pi/3)] \leqslant \epsilon \leqslant (2/3)\kappa\beta\cos\gamma \qquad (3.5)$$

Let us introduce the new parameter δ instead of ϵ, putting

$$\epsilon = (2/3)\kappa\beta\cos(\gamma + 2\delta), \qquad (3.6)$$

then the full variation of ϵ is attained when $0 \leqslant \delta \leqslant \pi/3$.

Note that the region of integration over (q, s) depends on the position of the straight line $\epsilon_\nu(q, s) = 0$ about vertex 3 (the shaded regions in Fig. 2). It can easily be seen that the condition of the straight line $\epsilon_\nu(q, s) = 0$ passing through vertex 3 may be written in the form $\gamma + \delta = \pi/3$. Thus, the computation of integrals over (q, s) should be carried out separately for $\gamma + \delta < \pi/3$ and $\gamma + \delta > \pi/3$. Simple calculations give in the case $\gamma + \delta < \pi/3$

$$\sum_\nu \rho_{\nu\nu} = 2\sqrt{3}\,\Omega \iint_{\epsilon_\nu<0} dqds = 6\,\Omega \frac{\sin\gamma}{\sin 3\gamma} \frac{\sin^2\delta \sin^2(\gamma + \delta)}{\sin^4(\pi/3)}. \qquad (3.7)$$

It is convenient to introduce for the shell occupation characteristic the quantity

$$\chi = 1 - N/\Omega, \qquad (3.8)$$

which at $0 < N < 2\Omega$ varies within the limits $-1 < \chi < 1$. The equation (2.7), in view of (3.7), leads to the following relation between δ and χ

$$1 - \chi = 6 \frac{\sin\gamma}{\sin 3\gamma} \frac{\sin^2\delta \sin^2(\gamma + \delta)}{\sin^4(\pi/3)}; \qquad (\gamma + \delta < \frac{\pi}{3}) \qquad (3.9)$$

Analogous calculations for $\gamma + \delta > \pi/3$ result in a similar formula, but with substitution in (3.9)

$$\chi \to -\chi$$

$$\gamma \to \bar{\gamma} \equiv (\pi/3) - \gamma; \quad \delta \to \bar{\delta} \equiv (\pi/3) - \delta. \qquad (3.10)$$

The right-hand part of Eq. (2.15) is calculated in a similar way

$$\sum_{\nu} \rho_{\nu\nu}(q_{\nu}\cos\gamma + s_{\nu}\sin\gamma) \to 2\sqrt{3}\,\Omega \iint_{\epsilon_{\nu}<0} dqds\ (q\cos\gamma + s\sin\gamma).$$

As a result, we get for $\gamma + \delta < \pi/3$

$$\beta = \frac{3}{2}\Omega\,\frac{\sin\gamma}{\sin 3\gamma}\,\frac{\sin^2\delta\sin^2(\gamma+\delta)}{\sin^4(\pi/3)}\left[\frac{\cos\delta\cos(\gamma+\delta)}{\cos^2(\pi/3)} - \frac{\sin\delta\sin(\gamma+\delta)}{\sin^2(\pi/3)}\right] \quad (3.11)$$

$$(\gamma + \delta < \frac{\pi}{3}).$$

Equation (3.9) defines parameter δ as a function of the occupation factor χ. Eliminating δ from (3.11) we get*

$$\beta = \Omega(1-\chi)\left[\cos\gamma - \sqrt{\frac{\sin 3\gamma}{6\sin\gamma}(1-\chi)}\right];\ (\gamma + \delta \leqslant \pi/3) \quad (3.12)$$

yielding the extremum value β at fixed γ*. The definition of the region in which (3.12) is valid after the exclusion of δ is as follows

$$\tan\gamma \leq \sqrt{3}\,[(1+\chi)/(3-\chi)]\ ;\ (\gamma + \delta \leqslant \pi/3). \quad (3.13)$$

In the region $\gamma + \delta > \pi/3$ the equation for β analogous to (3.12) is of the form (after substituting (3.10))

$$\beta = \Omega(1+\chi)\left[\cos\bar{\gamma} - \sqrt{\frac{\sin 3\bar{\gamma}}{6\sin\bar{\gamma}}(1+\chi)}\right]\ ;\ (\gamma + \delta > \pi/3). \quad (3.14)$$

It can easily be seen that expressions (3.12) and (3.14) coincide at the boundary $(\gamma + \delta = \pi/3)$. Note that equation (3.13) for the boundary of the two regions is invariant with respect to the replacement $\gamma \to \bar{\gamma}$; $\chi \to -\chi$.

Substituting the extremum values $\beta(\gamma)$ (3.12) or (3.14) in (2.10) we get the ground-state energy W as a function of parameter γ

$$W(\gamma) = \hbar\omega n N - \tfrac{1}{2}\kappa\beta^2(\gamma) - \tfrac{1}{2}\Delta^2/G. \quad (3.15)$$

In the region of large deformations considered the pairing energy Δ^2/G may be neglected compared with the deformation energy $\kappa\beta^2$, so that the absolute minimum of $W(\gamma)$ is defined by the maximum $\beta(\gamma)$. The investigation of function $\beta(\gamma)$ defined by Eqs. (3.12) and (3.14) is not very difficult. Let us state here the results.

The values $\gamma = 0$ and $\gamma = \pi/3$ are always extreme and one has to distinguish three regions of shell occupation:

(1) $-1 < \chi < -1/8$; $\gamma = 0$ - minimum;
$\gamma = \pi/3$ - maximum.
(2) $-1/8 < \chi < 1/8$; $\gamma = 0$ and $\gamma = \pi/3$ - maximum,

the absolute maximum is in $\gamma = \pi/3$ when $\chi < 0$ and in $\gamma = 0$ when $\chi > 0$. For

* When comparing the right side of (3.12) and general expression (2.27) we see that the term $C\beta_*$ in this case, is lacking. This is due to the fact that we consider nucleons on one shell only. Taking into account matrix elements of q_{μ} connecting shells n, n ± 2 will lead to the appearance of this term and to the renormalization of equilibrium deformation β (see 2.34).

$\chi = 0$ (half-filled shell) both maxima become equal. There exists an intermediate minimum for $\gamma = \gamma(\neq \chi)$.

(3) $1/8 < \chi < 1$; $\gamma = 0$ - maximum; $\gamma = \pi/3$ - minimum.

The form of function $W(\gamma)$ in the above three regions of χ is illustrated in Fig. 3.

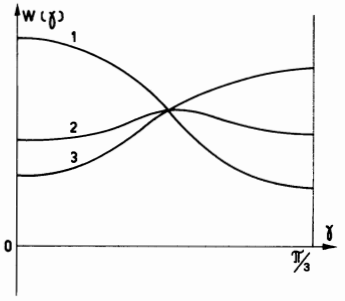

Fig. 3

Thus, the equilibrium form is always axially symmetrical. In filling the first half of the shell ($\chi > 0$) a prolate shape is advantageous ($\gamma = 0$), and in filling the second half ($\chi < 0$) it is oblate ($\gamma = \pi/3$).

3.2. Improved oscillator model

The total symmetry with respect to transformation from prolate to oblate shape ($\gamma \rightarrow \overline{\gamma}$) with simultaneous substitution of particles by holes ($\chi \rightarrow -\chi$) is a specific feature of the spherical harmonic oscillator model due to the additional degeneracy of single-particle levels in the orbital angular momentum. As is well known, good approximation to the real level scheme is attained by the inclusion of the term $D\ell^2$ yielding splitting over ℓ to the oscillator potential as well as spin-orbit interaction $C(\vec{\ell}.\vec{s})$ [7]. We shall consider these correction terms as perturbation. Such a consideration is valid, in any case, for the region of large deformations when level splitting caused by deformation is higher than that caused by terms $D\ell^2$ and $C(\vec{\ell}.\vec{s})$.

The correction of the first order to the ground state energy caused by the perturbation potential is given by the expression

$$W^{(1)} = \mathrm{Sp}\,(\hat{\rho}\hat{V}) = \sum_{\nu} \rho_{\nu\nu} \langle \nu | \hat{V} | \nu \rangle . \tag{3.16}$$

Let us consider term $D\ell^2$ for which the diagonal matrix element is given by

$$\begin{aligned} D\langle n_x\, n_y\, n_z | \ell^2 | n_x\, n_y\, n_z \rangle &= 2D(n_x\, n_y + n_y\, n_z + n_z\, n_x + n) \\ &= (3/2)\, Dn^2\, [4/9 - (q_\nu^2 + s_\nu^2)] \end{aligned} \tag{3.17}$$

where q and s have been defined in (3.2). Substituting (3.17) into (3.16) we find

$$W^{(1)} = (3/2)\, Dn^2 \sum_{\nu} \rho_{\nu\nu}\, [(4/9) - (q_\nu^2 + s_\nu^2)] - (1/4)\, Dn^4 . \tag{3.18}$$

For the larger symmetry of the formulas we have subtracted from $W^{(1)}$ the value $(1/4)Dn^4 = \frac{1}{2}\{W^{(1)}(N = 2\Omega) - W^{(1)}(N = 0)\}$. Using (3.4) and proceeding in

(3.18) to the integration, excluding δ by means of (3.9) we obtain in the region $\gamma + \delta < \pi/3$

$$W^{(1)} = -D\Omega^2 + \sqrt{\frac{32}{3}}\, D\Omega^2 (1-\chi)^{3/2} \sqrt{\frac{\sin\gamma}{\sin 3\gamma}} \times \left\{ \cos\gamma - \frac{3}{8} \left(1 + 2\frac{\sin\gamma}{\sin 3\gamma}\right) \sqrt{\frac{\sin 3\gamma}{6\sin\gamma}(1-\chi)} \right\} \tag{3.19}$$

and when $\gamma + \delta \geq \pi/3$:

$$W^{(1)} = D\Omega^2 - \sqrt{\frac{32}{3}}\, D\Omega^2 (1+\chi)^{3/2} \sqrt{\frac{\sin\bar{\gamma}}{\sin 3\bar{\gamma}}} \times \left\{ \cos\bar{\gamma} - \frac{3}{8} \left(1 + 2\frac{\sin\bar{\gamma}}{\sin 3\bar{\gamma}}\right) \sqrt{\frac{\sin 3\bar{\gamma}}{6\sin\bar{\gamma}}(1+\chi)} \right\}; \tag{3.20}$$

from (3.19) and (3.20) follows the relation

$$W^{(1)}(\chi, \gamma) = -W^{(1)}(-\chi, \bar{\gamma}) \tag{3.21}$$

which confirms the fact that the term $D\ell^2$ eliminates the symmetry with respect to the replacement $\gamma \to \bar{\gamma}$; $\chi \to -\chi$.

Function $W^{(1)}(\gamma)$ defined by (3.19) and (3.20) has the following specific features (parameter $D < 0$ [7]):

(1) $-1 < \chi < -\frac{1}{2}$. Minima when $\gamma = 0$ and $\pi/3$ intermediate maximum for $\gamma \neq 0, \pi/3$.

(2) $-\frac{1}{2} < \chi < \frac{1}{2}$. Minimum when $\gamma = 0$, maximum when $\gamma = \pi/3$.

(3) $\frac{1}{2} < \chi < 1$. Maxima when $\gamma = 0$ and $\gamma = \pi/3$ intermediate minimum for $\gamma \neq 0, \pi/3$ (see Fig. 4).

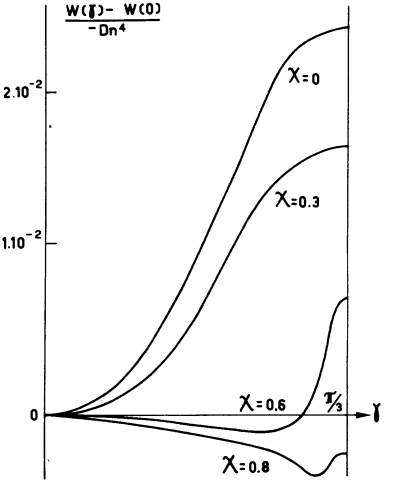

Fig. 4

It should be pointed out here that except for the narrow regions $\sim 0.75 < |\chi| < 1$ it holds everywhere that $W^{(1)}(\pi/3) > W^{(1)}(0)$.

As follows from the above analysis, the term $D\ell^2$ eliminates the symmetry between prolate and oblate deformations and expands the stability region of prolate nuclei. The minimum $W^{(1)}(\gamma)$ at the intermediate values of γ, as may be seen from the plots in Fig. 4, is hardly marked. It becomes notable only in the very beginning of shell filling ($\chi \approx 0.8 - 0.9$) where the value $W^{(1)}$ is itself small. Therefore, the term $D\ell^2$ may lead to the equilibrium non-axial deformation only when the ratio $|D|/\kappa$ is large enough. (At $\chi = 0.8$ it is necessary that $|D|/\kappa > 0.3$). According to the Nilsson model we put $|D| \approx 0.025\ \hbar\omega$. The parameter of quadrupole interaction estimated in [8] is $\kappa = 250\ A^{-1}$ MeV*. Then for the parameter ratio we find $|D|/\kappa \approx 0.1$ (for A = 125), which is too small for the appearance of equilibrium γ-deformation.

Thus, the main effect caused by the term $D\ell^2$ is the elimination of symmetry with respect to the oblate and prolate shapes. Figure 5 illustrates the energetical difference between the oblate and prolate states for various parameter values $|D|/\kappa$. As one may see from the plots, when $|D|/\kappa \approx 0.3$ the region of oblate deformations practically disappears.

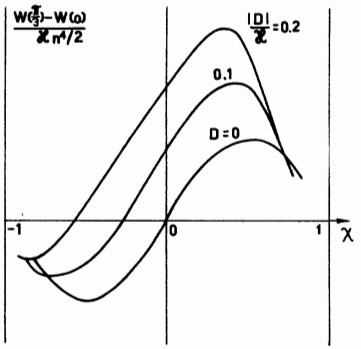

Fig. 5

Spin-orbit interaction $c(\vec{\ell}\cdot\vec{s})$ essentially changes the order of single-particle levels, but its contribution to the ground-state energy in the first order of perturbation theory is zero.

3.3. System with two types of nucleons

Till now we have been concerned with the system consisting of nucleons of a single type. Let us extend our results to the system of neutrons and protons. Cooper pairing occurs independently in neutron and proton subsystems. Quadrupole interaction should be considered identical for any pair of nucleons.** Then it can easily be seen that the basic formulae require only insignificant modifications. Equation (2.4) is valid if by Q_μ we mean the total quadrupole moment of neutrons and protons. Equations (2.7) and (2.8) are

* Our value κ differs from the parameter X in paper [8] by the factor 9/4, namely $\kappa = (9/4)\,X$.

** For the discussion of the possible difference in quadrupole interaction constants see [9].

now valid separately for neutrons and protons and define $\Delta_{n(p)}$ and $\lambda_{n(p)}$ respectively

$$\sum_{\nu} \rho^{n(p)}_{\nu\nu} = \tfrac{1}{2} \sum_{\nu_{n(p)}} \left(1 - \frac{\epsilon_\nu}{(\epsilon^2_\nu + \Delta^2_{n(p)})^{\frac{1}{2}}}\right) \equiv N_{n(p)}$$
$$G_{n(p)} \sum_{\nu_{n(p)}}^{\frac{1}{2}} \frac{1}{(\epsilon^2_\nu + \Delta^2_{n(p)})^{\frac{1}{2}}} = 1 \qquad (3.22)$$

The equation for Q_μ instead of (2.9) takes the form

$$Q_\mu = Q^n_\mu + Q^p_\mu = \sum_{\nu_p} \rho^n_{\nu\nu} \langle \nu | q_\mu | \nu \rangle + \sum_{\nu_p} \rho^p_{\nu\nu} \langle \nu | q_\mu | \nu \rangle \qquad (3.24)$$

and presently for the ground-state energy (2.10) we have

$$W(Q) = \sum_{\nu_n} \rho^n_{\nu\nu} (\epsilon_\nu + \lambda^n) + \sum_{\nu_p} \rho^p_{\nu\nu} (\epsilon_\nu + \lambda^p) + \tfrac{1}{2}\kappa \sum_\mu Q^*_\mu Q_\mu \qquad (3.25)$$
$$- \Delta^2_n / 2G_n - \Delta^2_p / 2G_p.$$

The treatment of the problem of nuclei transition from the spherical to the deformed shape has not essentially changed if we assume that the parameters (a, b, $B(\gamma)$, C) entering (2.36) and (2.37) are determined as some values averaged over neutrons and protons. Let us proceed, therefore, to the problem of equilibrium shape for the two-component nucleus in the region of large deformations (oscillator model).

As is directly seen from the structure of Eq. (3.24), equilibrium deformation β is determined by the expression

$$\beta(\gamma) = \beta_n(\gamma, \chi_n) + \beta_p(\gamma, \chi_p) \qquad (3.26)$$

where the function $\beta(\gamma, \chi)$ is given by the right sides of (3.12) and (3.14). The ground-state energy as a function of γ now takes the form of (cf. (3.15))

$$W(\gamma) = (\hbar\omega n N)_n + (\hbar\omega n N)_p - \tfrac{1}{2}\kappa\beta^2(\gamma) - \Delta^2_n / 2G_n - \Delta^2_p / 2G_p \qquad (3.27)$$

and (when neglecting pairing) is defined by the behaviour of $\beta(\gamma)$ (3.26). (We have not yet considered corrections to the oscillator model). The functions $\beta_{n(p)}(\gamma, \chi_{n(p)})$ (3.12), (3.14) in the first half of the shell ($\chi > 0$) are monotonously decreasing functions of γ, and in the second half monotonously increasing ones (except the narrow region of non-monotony near the middle of the shell). If neutrons and protons occupy the same halves of the shells then the sum of functions (3.26) will possess the same properties and, consequently, the equilibrium shape will be axially symmetrical. But if neutrons and protons occupy different halves of the shells ($\chi_n \chi_p < 0$) then function (3.26) may have a maximum for intermediate values of γ, thus leading to the non-axial equilibrium shape. We shall not carry out the cumbersome analysis of the two-parameter function (3.26) and find its maximum, restricting ourselves to estimating the parameter region only, at which there exists an intermediate maximum of $\beta(\gamma)$ (i.e. minimum $W(\gamma)$).
The function $W(\gamma)$ undoubtedly possesses a minimum for $0 < \gamma < \pi/3$ if the points $\gamma = 0$ and $\gamma = \pi/3$ (always extreme) correspond to maxima of $W(\gamma)$. Let us find, therefore, the region of χ_n and χ_p, for which $W(\gamma)$ possesses a maximum when $\gamma = 0, \pi/3$. These regions, whose boundaries are defined by means of (3.27), (3.12) and (3.14) from the equations

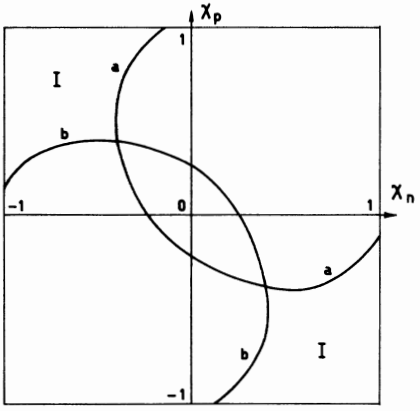

Fig. 6

$$[(1/\gamma)(dW/d\gamma)]_{0=\gamma} = 0 \; ; \left[(1/\bar{\gamma})(dW/d\bar{\gamma})\right]_{\bar{\gamma}=0} = 0$$

are illustrated in Fig. 6. As can be seen from the plots, there in fact exist such values of χ_n, χ_p for which the function $W(\gamma)$ possesses a maximum both for $\gamma = 0$ and $\gamma = \bar{\gamma}/3$ (region I limited by curves a, b). Such a case takes place when one sort of nucleon occupies the first third of the shell ($1 > \chi > 0.4$ or $0 < N/2\Omega < 0.3$), and the second occupies the last third ($-0.4 > \chi > -1$ or $1 > N/2\Omega > 0.7$). It should be noted that such an occupation of neutron and proton shells is not likely to occur in the known region of the deformed nuclei. The taking into account of term $D\ell^2$ leads to an inessential variation of the region of χ_n, χ_p with an intermediate minimum of $W(\gamma)$. Thus, in the two-component system, too, the appearance of an axially non-symmetrical equilibrium shape is in practice hardly probable.

It is clear that we cannot expect a death sentence for the γ-deformation from the qualitative analysis given above. It would be desirable to perform quantitative calculations using the real scheme of one-particle levels for those regions where the appearance of stable γ-deformation is most probable.

4. THE ROTATION OF DEFORMED NUCLEI. MOMENT OF INERTIA

4.1. Formulation of the problem

Let us proceed now to the consideration of collective excitations of nuclei, i.e. to the problem of the time-dependent self-consistent field. First, we shall consider the case of rotation of the deformed nuclei. This question refers in fact to the problem of finding the time-dependent self-consistent field, but the process of its solutions involves great difficulties. The so-called cranking-model has been generally used (or other equivalent methods) and results in the apparently correct value of the moment of inertia in case of rotation. Rotation of the quantum system leads to the additional energy

$$E_J^{rot} = \frac{\hbar^2 J(J+1)}{2\mathcal{J}} \qquad (4.1)$$

where J is the quantum number of the angular momentum and $\mathcal{J}$ the moment of inertia of the system. The separation of terms of the type of (4.1) from the Hamiltonian of the nucleon system, using a reasonable approximation, is a very complicated problem. The main complication is that perturbation theory must be constructed so that the rule for angular momenta addition (algebraically non-additive preservation law) is fulfilled. This difficulty may be overcome in the following way. Let us consider a rotational state with very large J, then the rotation is quasi-classical, and we may speak of the definite direction of the rotation axis (x) and instead of (4.1) we can write, assuming $J \approx J_x$,

$$E_J^{rot} \approx \hbar^2 J_x^2 / 2\mathcal{J} = \tfrac{1}{2}\mathcal{J}\,\Omega_x^2 \qquad (4.2)$$

where $\Omega_x = \hbar J_x / \mathcal{J}$ is the angular velocity of rotation. Thus at large J the problem is reduced to the finding of the energy of state with a definite angular velocity. But since the moment of inertia is a coefficient independent of J, and having found it from (4.2), we simultaneously also define (4.1). The substitution of (4.1) by (4.2) is the main idea of the cranking model. We make use of an equivalent method.

We shall look for the lowest state of the system at a fixed average value of the angular momentum about the axis of rotation (perpendicular to the nucleus symmetry axis). For this purpose we shall add the term

$$H_\Omega = -\Omega \hat{J}_x = -\Omega \sum_{12} \langle 1 | j_x | 2 \rangle a_1^+ a_2 . \qquad (4.3)$$

to the Hamiltonian, where $\hat{J}_x$ is the operator of angular momentum and Ω is the Lagrangian multiplier which will be determined later from the condition $\langle \hat{J}_x \rangle = J_x$.

4.2. General expression for moment of inertia

Let us consider (4.3) as perturbation. Then the problem is reduced to the calculation of the correction to the ground state of the system under external perturbation. Since the angular momentum operator changes its sign under time reversal, i.e.

$$\langle 1 | j_x | 2 \rangle = -\langle \tilde{2} | j_x | \tilde{1} \rangle, \qquad (4.4)$$

then, according to the classification of perturbation (1.21), we have in this case

$$V_{12}^{(-)} = -\Omega \langle 1 | j_x | 2 \rangle ; \; V_{12}^{(+)} = 0; \qquad (4.5)$$

therefore, we may put $Z_{12}^{(+)} = 0$. To define $Z^{(-)}$ we have from (1.25), (1.21) and (4.5)

$$E_{12} Z_{12}^{(-)} - \xi_{12}^{(-)} \Delta_{12}^{(-)} + \eta_{12}^{(-)} U_{12}^{(-)} = \Omega\, \eta_{12}^{(-)} \langle 1 | j_x | 2 \rangle . \qquad (4.6)$$

The correction to the ground-state energy according to (1.31) is, in this case, equal to

$$\langle H \rangle^{(2)} = \tfrac{1}{2} \Omega^2 \sum_{12} \eta_{12}^{(-)} \langle 1 | j_x | 2 \rangle \, (Z_{12}^{(-)*} / \Omega). \qquad (4.7)$$

The parameter Ω is found from the equation $\langle \hat{J}_x \rangle = J_x$, i.e. in view of (4.3) and (1.32)

$$\langle H_\Omega \rangle = -\Omega^2 \sum_{12} \eta_{12}^{(-)} \langle 1 | j_x | 2 \rangle (Z_{12}^{(-)*}/\Omega) = -\Omega J_x. \tag{4.8}$$

From (4.7) and (4.8) it follows (see (4.2)) that

$$\langle H \rangle^{(2)} = \tfrac{1}{2} J_x^2 / \mathcal{J}, \tag{4.9}$$

where the moment of inertia $\mathcal{J}$ is determined by the following expression

$$\mathcal{J} = \sum_{12} \eta_{12}^{(-)} \langle 2 | j_x | 1 \rangle (Z_{12}^{(-)}/\Omega). \tag{4.10}$$

For analysis, it would be convenient to represent $\mathcal{J}$ in separate terms. Using (4.6) we write

$$\mathcal{J} = \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)} \tag{4.11}$$

where

$$\mathcal{J}^{(1)} = \sum_{12} |\langle 1 | j_x | 2 \rangle|^2 (\eta_{12}^{(-)2}/E_{12}) \tag{4.12}$$

$$\mathcal{J}^{(2)} = \sum_{12} \langle 2 | j_x | 1 \rangle (\Delta_{12}^{(-)}/\Omega)(\xi_{12}^{(-)} \eta_{12}^{(-)}/E_{12}) \tag{4.13}$$

$$\mathcal{J}^{(3)} = -\sum_{12} \langle 2 | j_x | 1 \rangle (U_{12}^{(-)}/\Omega)(\eta_{12}^{(-)2}/E_{12}). \tag{4.14}$$

The consideration of the rotation by means of the adiabatic perturbation theory (the usual form of the cranking model) gives only the first term in the moment of inertia [5]. This method is equivalent to taking into account the diagonal corrections to the average values only, i.e. $\langle a_1^+ a_1 \rangle^{(2)}$ and $\langle a_{\bar{1}} a_2 \rangle^{(2)}$ which may be done within the framework of the usual Bogolyubov u, v- transformation. The consideration of non-diagonal averages results in two additional terms. The term $\mathcal{J}^{(2)}$ takes account of the effect of rotation on Cooper pairing* (this is expressed by the appearance of $\Delta_{12}^{(-)}$). The last term, as follows from its structure, describes the variation of the self-consistent field under rotation. Note that so far we have obtained the explicit expression only for the main term $\mathcal{J}^{(1)}$ (4.12). Expressions (4.13) and (4.14) contain the function $Z_{12}^{(-)}$, which must be found in order to solve integral equation (4.6).

4.3. Estimation of the basic term $\mathcal{J}^{(1)}$

For quantitative calculations of the moment of inertia for real even-even nuclei we may restrict ourselves to the term $\mathcal{J}^{(1)}$. Using the explicit expression for $\eta^{(-)2}$ (see second footnote of 1.4) we have from (4.12) (considering $\Delta^{(0)}$ to be constant)

$$\mathcal{J}^{(1)} = \sum_{12} |\langle 1 | j_x | 2 \rangle|^2 [E_1 E_2 - \epsilon_1 \epsilon_2 - \Delta^2]/ 2E_1 E_2 (E_1 + E_2). \tag{4.15}$$

To estimate (4.15) let us use the method proposed in [6]. At the fixed index 1

* The term $\mathcal{J}^{(2)}$ was first obtained by MIGDAL, who used the method of Green's functions [6].

in the sum of (4.15) the second index 2 takes just the few values permitted by the selection rules for the matrix element of j_x. As can easily be checked the last factor in the sum at fixed difference $\epsilon_1 - \epsilon_2$ has an abrupt maximum over variable ϵ_1 whose width is Δ (or $|\epsilon_1 - \epsilon_2|$ if $|\epsilon_1 - \epsilon_2| > \Delta$) at the Fermi surface and

$$\int d\epsilon_1 [E_1 E_2 - \epsilon_1 \epsilon_2 - \Delta^2] / 2E_1 E_2 (E_1 + E_2) = 1 - g[(\epsilon_1 - \epsilon_2)/2\Delta] \qquad (4.16)$$

where

$$g(x) = \frac{\operatorname{arsh} x}{(1 + x^2)^{1/2}}. \qquad (4.17)$$

If there is a sufficient number of levels in the region Δ (or $|\epsilon_1 - \epsilon_2|$) (this condition is practically valid for nuclei) one may make the following replacement in (4.15)

$$\eta_{12}^{(-)2} = [E_1 E_2 - \epsilon_1 \epsilon_2 - \Delta^2] / 2E_1 E_2 (E_1 + E_2) \to \left[1 - g\left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)\right] \delta(\epsilon_1). \qquad (4.18)$$

As a result we get

$$\mathcal{J}^{(1)} = \sum_{12} |\langle 1 | j_x | 2 \rangle|^2 \left[1 - g\left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)\right] \delta(\epsilon_1). \qquad (4.19)$$

In the model of the oscillator potential the matrix element $\langle 1 | j_x | 2 \rangle$ for the two types of transitions is different from zero:

(1) Transitions inside a single oscillator shell ("near transitions") for which

$$d' = |\epsilon_1 - \epsilon_2| = \hbar |\omega_\perp - \omega_z| \approx \hbar \omega \, \delta R/R \qquad (4.20)$$

(here ω_z, $\omega_\perp$ are oscillator frequencies in different directions, and $\delta R/R$ is the deformation of the potential);

(2) Transitions across the shell ("distant transitions") for which

$$d'' = |\epsilon_1 - \epsilon_2| \approx 2\hbar\omega. \qquad (4.20')$$

The following inequalities are then valid for near and distant transitions

$$d'/2\Delta \lesssim 1; \quad d''/2\Delta \gg 1. \qquad (4.20'')$$

The matrix elements of transitions are inversely proportional to the corresponding values of d. Taking account of all the above, we get the following estimation for (4.19)

$$\mathcal{J}^{(1)} \approx \mathcal{J}_0 [1 - g(d'/2\Delta)], \qquad (4.21)$$

where the rigid body moment of inertia is denoted by $\mathcal{J}_0$, which may be formally obtained from (4.15) or (4.19) in the limit $\Delta \to 0$. As follows from (4.21), Cooper pairing considerably decreases the moment of inertia. Although the above evaluations refer to the oscillator potential, qualitatively, they are also valid for other models.

A number of authors [10, 11] have carried out detailed calculations of the moment of inertia using formula (4.15) on the basis of the semi-empirical

Nilsson level scheme [7]. The results for all nuclei are in good agreement with the experimental values of $\mathcal{J}$ (deviations, as a rule, are less than 10%).

4.4. Hydrodynamical limit and gauge invariance

From a purely theoretical viewpoint one should not be restricted only to the term $\mathcal{J}^{(1)}$ in the moment of inertia in spite of the good agreement with experiment. In fact, in the limits of a very strong pairing strength ($\Delta \gg d', d''$), when the size of the Cooper pair $r_0 \sim \hbar v/\Delta$ (v - velocity at Fermi surface) becomes much less than the nuclear radius, the hydrodynamical equations of an ideal liquid should be valid, resulting in a moment of inertia

$$\mathcal{J}_{\text{irrot}} \approx \mathcal{J}_0 \, (\delta R/R)^2 . \tag{4.22}$$

Expression (4.15) or (4.19) in this limit becomes zero. The main role in this case belongs to the term $\mathcal{J}^{(2)}$ which yields (4.22)*.

Note that if interaction between nucleons is approximated by the pairing interaction (2.6), then $\Delta_{12}^{(-)} = 0$ and the term $\mathcal{J}^{(2)}$ becomes zero. Thus, when using model interaction (2.6), the transition to hydrodynamics at $\Delta \to \infty$ is lacking. This is associated with the fact that interaction (2.6) is gauge non-invariant. For gauge-invariant interaction the "diagonal" matrix elements $\langle 1\tilde{1}|G|\tilde{1}'1'\rangle$ are connected with the "non-diagonal" $\langle 1\tilde{2}|G|\tilde{2}'1'\rangle$, so that the latter cannot be randomly considered to be equal to zero, as in (2.6). Let us consider, for instance, central short-range forces

$$G(\vec{r}_1 \vec{r}_2) = -G\delta(\vec{r}_1 - \vec{r}_2). \tag{4.23}$$

In this case equation (1.19″) for Δ can be transformed to

$$G \sum_1 \frac{\varphi_1^*(\vec{r})\,\varphi_1(\vec{r})}{2E_1} = 1. \tag{4.24}$$

For an arbitrary function of the space coordinates $\phi(\vec{r})$, in view of (4.23) and (4.24), one may easily get the following equation

$$\phi_{12}\,\gamma_2 = -\sum_{1'2'} \langle 1\tilde{2}|G|\tilde{2}'1'\rangle (E_{1'2'}/4E_{1'}E_{2'}) \phi_{1'2'}\gamma_{2'} , \tag{4.25}$$

yielding the relation between the different matrix elements of the interaction. (The value $\gamma_2 = -\gamma_{\tilde{2}}$ has been defined in (1.19‴)).

The connection between the gauge-invariance of the interaction and the possibility of passing over to hydrodynamics has a simple physical meaning. In the hydrodynamic description macroscopically small parts of the liquid move relatively to each other in such a way that their intrinsic structure stays the same. In particular, the interaction between particles does not change. In other words a local (in the macroscopic sense) Galileian invariance exists in the liquid.

In a Galileian transformation the single-particle wave function transforms according to

$$\Psi(\vec{r}) \to \Psi(\vec{r})\, e^{i\vec{p}\cdot\vec{r}}, \tag{4.25′}$$

* This was shown by MIGDAL [6] who also obtained the explicit expression for $\mathcal{J}^{(2)}$ in an oscillator model.

where $\vec{P}/m$ is the relative velocity of the two frames of reference. If we have local Galileian invariance in the system, then $\vec{P}$ is a function of the space coordinates and hence the transformation (4.25') actually coincides with the gauge transformation

$$\Psi(\vec{r}) \to \Psi(\vec{r})\, e^{i\chi(\vec{r})} \qquad (4.25'')$$

where $\chi(\vec{r})$ is an arbitrary function of the space coordinates. Thus the interaction matrix element $\langle 1\tilde{2}|G|\tilde{2}'1'\rangle$ in a hydrodynamical system must be invariant under transformations (4.25'').

The model Hamiltonian (2.6) is rather poor in the sense that it only describes the interaction of pairs at rest.* When the pair moves as a whole the interaction is switched off. Naturally it is not possible to pass over to the hydrodynamic limit for an arbitrarily large interaction of such a type.

4.5. Estimation of the term $\mathcal{I}^{(2)}$

Let us transform the integral equation (4.6) making use of (4.25). For simplicity let us neglect the variation of the self-consistent field under rotation and assume $U_{12}^{(-)} = 0$. Then, using the definition of $\Delta^{(-)}$ (1.36), we get from (4.6)

$$\Delta_{12}^{(-)} + \sum_{1'2'}\langle 1\tilde{2}|G|\tilde{2}'1'\rangle\,(\xi_{1'2'}^{(-)2}/E_{1'2'})\Delta_{1'2'}^{(-)} \qquad (4.26)$$

$$= -\Omega\sum_{1'2'}\langle 1\tilde{2}|G|\tilde{2}'1'\rangle\,(\xi_{1'2'}^{(-)}\eta_{1'2'}^{(-)}/E_{1'2'})\,\langle 1'|j_x|\tilde{2}\rangle.$$

In the case of central interaction the value $\Delta^{(-)}$ is a function of the space coordinates, so that for $\Delta_{12}^{(-)}$ ($\to \gamma_2\,\phi_{12}$) the equality (4.25) is valid. As a result, equation (4.26) may be reduced to the form

$$\sum_{1'2'}\langle 1\tilde{2}|G|\tilde{2}'1'\rangle\left\{\left(\frac{E_{1'2'}}{4E_1E_{2'}} - \frac{\xi_{1'2'}^{(-)2}}{E_{1'2'}}\right)\Delta_{1'2'}^{(-)} - \Omega\frac{\xi_{1'2'}^{(-)}\eta_{1'2'}^{(-)}}{E_{1'2'}}\langle 1'|j_x|2'\rangle\right\} = 0$$

from which, after substituting $\xi^{(-)}, \eta^{(-)}$ and using (4.23), we obtain

$$\sum_{12}\varphi_2^*(\vec{r})\,\varphi_1(\vec{r})\,[1/E_1E_2(E_1+E_2)]\,\{(\epsilon_1-\epsilon_2)^2\gamma_2\Delta_{12}^{(-)} - 2\Omega\Delta\,(\epsilon_1-\epsilon_2)\langle 1|j_x|2\rangle\} = 0. \qquad (4.27)$$

Note that $i(\epsilon_1-\epsilon_2)\langle 1|j_x|2\rangle \equiv j_{12}^x$ is the matrix element of a function of the space coordinates $j_{12}^x = y(\partial u/\partial z) - z(\partial u/\partial y)$. Therefore, multiplying (4.27) by $j_{12}^x(\vec{r})$ and integrating over $\vec{r}$, we obtain

$$\sum_{12}[1/E_1E_2(E_1+E_2)]\,\{(\epsilon_1-\epsilon_2)^2 j_{21}^x\, i\Delta_{12}^{(-)}\gamma_2 - 2\Omega\Delta|j_{12}^x|^2\} = 0. \qquad (4.28)$$

From (4.28) it is clear that equation (4.28) is satisfied if we assume that

$$\Delta_{12}^{(-)}\gamma_2 = -i\Omega\, j_{12}^x\, D/2\Delta \qquad (4.29)$$

---

* In a macroscopic system the above statement follows directly from the form of the interaction, analogous to (2.6), namely $\langle p - p|G|-p'\,p'\rangle$.

where

$$D = \sum_{12} \frac{|\overset{\bullet}{j}{}^{x}_{12}|^2}{E_1 E_2 (E_1 + E_2)} \Big/ \sum_{12} \frac{|\overset{\bullet}{j}{}^{x}_{12}|^2}{E_1 E_2 (E_1 + E_2)} \left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)^2. \tag{4.30}$$

From (4.13) we then find

$$\mathcal{I}^{(2)} = \left(\sum_{12} \frac{|\overset{\bullet}{j}{}^{x}_{12}|^2}{2E_1 E_2 (E_1 + E_2)}\right)^2 \Big/ \sum_{12} \frac{|\overset{\bullet}{j}{}^{x}_{12}|^2}{2E_1 E_2 (E_1 + E_2)} \left(\frac{\epsilon_1 - \epsilon_2}{2\Delta}\right)^2. \tag{4.31}$$

Let us now assume the self-consistent potential of the deformed nucleus to be of the form

$$U = U(r_\delta^2) \tag{4.32}$$

where

$$r_\delta^2 = (x^2 + y^2)/(1 - \tfrac{2}{3}\delta R/R) + z^2/[1 + (4/3)(\delta R/R)]. \tag{4.33}$$

Then for the function $\overset{\bullet}{j}{}^{x}$ we obtain (accurate up to terms linear in $\delta R/R$)

$$\overset{\bullet}{j}{}^{x} = y(\partial U/\partial z) - z(\partial U/\partial y) = -4yzU'(r^2)(\delta R/R). \tag{4.34}$$

Let us note also that in the case when $\Delta \gg d', d''$ one may substitute in (4.31)

$$1/E_1 E_2 (E_1 + E_2) \to (1/\Delta^2)\delta(\epsilon_1). \tag{4.35}$$

With the aid of (4.34) and (4.35) it is easy to see that the right side of (4.31) tends to the finite limit when $\Delta \to \infty$ and $\mathcal{I}^{(2)} \sim (\delta R/R)^2$ as it should be in the hydrodynamic limit.

5. VIBRATIONS OF DEFORMED NUCLEI

5.1. Model with quadrupole interaction

Let us apply the equations of the time-dependent self-consistent field to the consideration of the eigenvibrations of the deformed nuclei. We shall restrict ourselves to the consideration of quadrupole-type vibrations of nuclei with axially symmetrical deformations. Two types of vibrations are known to exist in this case. One type conserves the axial symmetry, β-vibrations, and in the second type the axial symmetry is disturbed, γ-vibrations. To solve the system of homogeneous, integral equations (1.35) we make a simplifying assumption as to the interaction matrix elements. To start with, we consider only the quadrupole interaction (2.1'), which naturally plays a decisive role in vibrations of the type considered. Then in (1.35) one must put

$$\Delta^{(\pm)}_{12} = \epsilon^{(-)}_{12} = 0, \tag{5.1}$$

$$\epsilon^{(+)}_{12} = -\kappa \sum_\mu q^*_\mu(12)\, Q_\mu,$$

where

$$Q_\mu = \sum_{12} q_\mu(12)\ \eta_{12}^{(+)} Z_{12}^{(+)}. \tag{5.2}$$

After substituting (5.1) into (1.35) we get

$$\omega Z_{12}^{(-)} = E_{12} Z_{12}^{(+)} - \kappa \eta_{12}^{(+)} \sum_\mu q_\mu^*(12)\, Q_\mu \tag{5.3}$$

$$\omega Z_{12}^{(+)} = E_{12}\, Z_{12}^{(-)}$$

or after eliminating $Z^{(-)}$

$$(E_{12}^2 - \omega^2) Z_{12}^{(+)} = \kappa \eta_{12}^{(+)} E_{12} \sum_\mu q_\mu^*(12)\, Q_\mu. \tag{5.4}$$

From (5.4) and (5.2) we obtain the system of algebraic equations for Q_μ

$$Q_\mu = \kappa \sum_{12} [E_{12} \eta_{12}^{(+)2} / (E_{12}^2 - \omega^2)]\, q_\mu(12) \sum_{\mu'} q_{\mu'}^*(12)\, Q_{\mu'}. \tag{5.5}$$

The factor $E_{12}\eta_{12}^{(+)2}/(E_{12}^2 - \omega^2)$ in the sum in (5.5) is a positive function with maximum at the Fermi surface while $q_\mu(12)$ is the alternating function. Hence, at not very large deformations ($\delta R / R \ll 1$) one may consider that the main term on the right-hand side of (5.5) is a term with $\mu' = \mu$. In this approximation we have

$$Q_\mu = \kappa \sum_{12} [E_{12} \eta_{12}^{(+)2} / (E_{12}^2 - \omega^2)]\, |q_\mu(12)|^2\, Q_\mu, \tag{5.6}$$

from which we get the following dispersion equation for eigenfrequency ω

$$\kappa \sum_{12} |q_\mu(12)|^2 [E_{12} \eta_{12}^{(+)2} / (E_{12}^2 - \omega^2)] = 1. \tag{5.7}$$

It is convenient to transform Eq. (5.7) somewhat. We introduce the notation

$$\sigma = \kappa \sum_{12} |q_\mu(12)|^2\ (\eta_{12}^{(+)2} / E_{12}) \tag{5.8}$$

$$B(\omega) = (\kappa/\sigma) \sum_{12} |q_\mu(12)|^2\ (\eta_{12}^{(+)2}/E_{12})\ [4\Delta^2/(E_{12}^2 - \omega^2)]. \tag{5.9}$$

Then Eq. (5.7) can be rewritten in the form

$$(\omega/2\Delta)^2 = [1/\sigma - 1]/B(\omega). \tag{5.10}$$

The right side of (5.10) has a simple physical meaning. The quantity $(1/\sigma - 1)$ is the restoring force coefficient (cf.(2.13)) and $B(\omega)$ the inertial parameter for the vibrations in question.

We shall use the same procedure in considering σ and $B(\omega)$ as when calculating the momentum of inertia. If we keep the difference $d = \epsilon_1 - \epsilon_2$ fixed, the quantity $\eta_{12}^{(+)2}/E_{12}$ as a function of ϵ_1 has a sharp maximum of the width Δ (or d if $d > \Delta$) at the Fermi surface. Assuming that $|q_\mu(12)|^2$ changes little in this interval*, and taking into account that the integral of $\eta_{12}^{(+)2}/E_{12}$ over ϵ_1 is equal to unity, we may make a replacement in (5.8):

$$\eta_{12}^{(+)2}/E_{12} \to \delta(\epsilon_1) . \tag{5.11}$$

The same arguments allow an analogous substitution in (5.9)

$$\frac{4\Delta^2 \eta_{12}^{(+)2}}{E_{12}(E_{12}^2 - \omega^2)} \to \frac{1 - g(\ell)}{\ell^2} \delta(\epsilon_1) \tag{5.12}$$

where g(x) has been defined in (4.17) and

$$\ell = \sqrt{(d^2 - \omega^2)/4\Delta^2} . \tag{5.13}$$

The quantity ℓ becomes purely imaginary for $\omega^2 > d^2 = (\epsilon_1 - \epsilon_2)^2$. For imaginary arguments the function g(x) has the form

$$g(ix) = \begin{cases} \dfrac{\arcsin x}{x\sqrt{1-x^2}} ; & (x^2 < 1) \\ -\dfrac{\operatorname{arch} x}{x\sqrt{x^2-1}} ; & (x^2 > 1) . \end{cases} \tag{5.14}$$

It can be seen from (5.14) that g(ix) has a singularity when x tends to unity from below.

Substituting (5.13) and (5.14) in (5.8) and (5.9) we obtain

$$\sigma = \kappa \sum_{12} \delta(\epsilon_1) |q_\mu(12)|^2 \tag{5.15}$$

$$B(\omega) = (\kappa/\sigma) \sum \delta(\epsilon_1) |q_\mu(12)|^2 (1 - g(\ell))/\ell^2 . \tag{5.16}$$

The quantity (5.16) can be expressed as

$$B(\omega) = [(1 - g(\ell))/\ell^2] \tag{5.17}$$

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* Otherwise it is necessary to consider values of $|q_\mu(12)|$ averaged over the interval Δ (or d).

where the symbol [ ] means averaging with the weight $\kappa\delta(\epsilon_1)|q_\mu(12)|^2$, i.e.

$$[X] = (\kappa/\sigma)\sum_{12}\delta(\epsilon_1)|q_\mu(12)|^2 X(\epsilon_1 - \epsilon_2). \tag{5.18}$$

The operator of the single-particle quadrupole moment q has two types of matrix elements: the first, with $d < 2\Delta$, connects states with near-lying energies (for q_0 diagonal); the second connects distant states with the transition energy $d'' \gg 2\Delta$. The weight function in (5.18) has consequently two peaks. Therefore when estimating the average (5.18) of a smooth function $X(\epsilon_1 - \epsilon_2)$ it can be assumed that

$$[X] = (\sigma'/\sigma) X(d') + (\sigma''/\sigma) X(d'') \tag{5.19}$$

where d' and d'' are mean values of $\epsilon_1 - \epsilon_2$ for near and distant transitions and σ', σ'' are the corresponding contributions of these transitions to (5.15).

Using (5.17) and (5.19) we can transform Eq. (5.10) to the form

$$F(\omega) \equiv (\omega/2\Delta)^2\left\{(\sigma'/\sigma)\frac{1-g(\ell')}{\ell'^2} + (\sigma''/\sigma)\frac{1-g(\ell'')}{\ell''^2}\right\} = \sigma^{-1} - 1. \tag{5.20}$$

The left side of (5.20), F, as a function of ω is shown schematically in Fig. 7. The solutions of the dispersion equation (5.20) are obtained as the intersec-

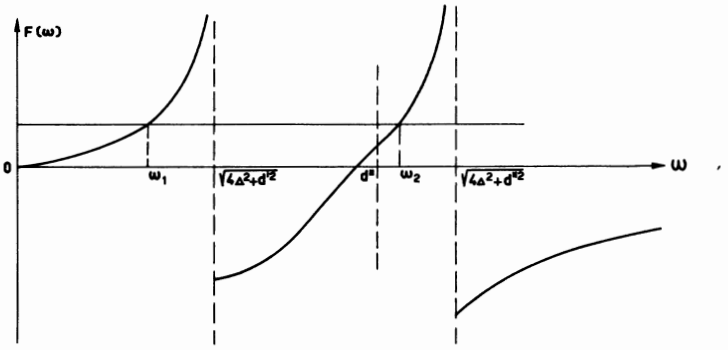

Fig. 7

tion points of the curve $F = F(\omega)$ with the horizontal line $F = \sigma^{-1} - 1$. It follows from the graph that there are two solutions

$$0 < \omega_1 < \sqrt{4\Delta^2 + d'^2}, \qquad \sqrt{4\Delta^2 + d'^2} \ll \omega_2 < \sqrt{4\Delta^2 + d''^2}. \tag{5.21}$$

Now we obtain these solutions directly from equation (5.10). Let us first consider the low-energy solution ω_1. Since $\omega_1 \ll d''$ we have $\ell'' \approx d''/2\Delta \gg 1$

and hence we can neglect the contribution of distant transitions in the sum in (5.16), containing the factor $(1 - g(\ell))/\ell^2$. If we now introduce the renormalized constant of the quadrupole interaction

$$\kappa_{\text{eff}} = \kappa/(1 - \sigma'') = \kappa/[1 - \kappa \sum_{12}'' \delta(\epsilon_1) |q_\mu(12)|^2] \tag{5.22}$$

the dispersion equation (5.10) can be written as

$$(\omega/2\Delta)^2 = \frac{1 - \kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2}{\kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2 (1-g(\ell))/\ell^2}, \tag{5.23}$$

i.e. it contains sums only over near transitions. Therefore mainly nucleons at the Fermi surface are involved in such excitations. The polarization of the other nucleons (described by distant transitions) only leads to a renormalization of the quadrupole interaction between the nucleons near the Fermi surface.

If the vibrations are adiabatic, i.e. if $\omega \ll 2\Delta$ (or, more exactly, if $|\ell'| \ll 1$), the expansion $g(\ell') \approx 1 - \frac{2}{3}\ell'^2$ can be used and (5.23) then becomes

$$(\omega/2\Delta)^2 = (3/2)\frac{1 - \kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2}{\kappa_{\text{eff}} \sum_{12}' \delta(\epsilon_1) |q_\mu(12)|^2}. \tag{5.24}$$

Note that the energies of the β and γ-vibrations (related to q_0 and q_2 respectively) do not differ in the adiabatic approximation, since the sums in (5.24) can easily be shown to be independent of μ. The more precise formula (5.23) gives a difference between ω_β and ω_γ and the inequality $\omega_\beta < \omega_\gamma$ always holds. This follows from the fact that for β-vibrations $q_0(12) \sim \delta_{12}$, i.e. $d' = 0$, whereas for γ vibrations $d' \neq 0$.*

Let us now consider the second solution (5.21). In this case $\omega_2 \sim d'' \gg 2\Delta$, i.e. $|\ell'| \sim 1$ and $|\ell'| \approx \omega_2/2\Delta \gg 1$. Therefore the contribution of the near transitions to the function $\sigma B(\omega)$ (5.16) is negligible. The solution ω_2 of Eq. (5.10) lies in the region

$$d'' < \omega_2 < \sqrt{4\Delta^2 + d''^2} \tag{5.25}$$

since it is just in this region that the function $(\omega/2\Delta)^2 B(\omega)$ rapidly increases from the value $(2\sigma''/3\sigma)(d''/2\Delta)^2$ to infinity (see Fig. 7).

Thus the second solution, lying in the region (5.25) describes vibrations of a large number of the nucleons in the nucleus, since it is substantially

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* Similar results have been obtained in [16, 17]. γ-transitions have also been studied in [9]. Comparisons with experimental data are given in [9] and [17].

determined by distant transitions. The energies of these vibrations exceed the single-particle excitation energy 2Δ and therefore the existence of real excitations of this type is rather doubtful.

5.2. Effect of the gauge-invariant pairing interaction

We have investigated eigenoscillations of deformed nuclei, considering, in fact, only two parts of the interaction: quadrupole interaction (2.1′) and pairing interaction (2.6). As we have seen in the case of rotation, the use of gauge non-invariant interaction (2.6) results in the loss of the term $\mathcal{J}^{(2)}$ in the moment of inertia. So it is quite natural in the case of vibrations, too, to consider the more realistic interaction (4.23) instead of (2.6).

It is convenient to proceed from equations (1.35) to the system of equations for $\Delta^{(\pm)}$ and $\epsilon^{(\pm)}$. Making use of (1.36) it is easy to get

$$\Delta_{12}^{(\pm)} + \sum_{1'2'} \frac{\langle 1\tilde{2}|G|\tilde{2}'1'\rangle}{E_{1'2'}^2 - \omega^2} \{ E\,\xi^{(\pm)2}\,\Delta^{(\pm)} - E\,\xi^{(\pm)}\eta^{(\pm)}\,\epsilon^{(\pm)} + \omega\,\xi^{(\mp)}\xi^{(\pm)}\Delta^{(\mp)} - \omega\,\xi^{(\pm)}\,\eta^{(\mp)}\,\epsilon^{(\mp)} \}_{1'2'} = 0$$

$$\mp\epsilon_{12}^{(\pm)} + \sum_{1'2'} \frac{\overline{\langle 1\tilde{1}'|G|\tilde{2}'2\rangle}}{E_{1'2'}^2 - \omega^2} \{ E\eta^{(\pm)}\xi^{(\pm)}\Delta^{(\pm)} - E\eta^{(\pm)2}\epsilon^{(\pm)} + \omega\,\eta^{(\pm)}\,\xi^{(\mp)}\Delta^{(\mp)} - \omega\,\eta^{(\pm)}\,\eta^{(\mp)}\,\epsilon^{(\mp)} \}_{1'2'} = 0 . \tag{5.26}$$

For simplification of equations (5.26) note that $\xi_{12}^{(+)}$ at fixed difference $\epsilon_1 - \epsilon_2$ is an odd function of $\epsilon_1 + \epsilon_2$, i.e. $\xi_{12}^{(+)}$ changes sign at the Fermi surface, while $\xi_{12}^{(-)}$, $\eta_{12}^{(\pm)}$, E_{12} are even functions of $\epsilon_1 + \epsilon_2$. Thus, one may neglect in (5.26) terms containing $\xi_{12}^{(+)}$ linearly. In this case the equation for $\Delta^{(+)}$ is separated. Further, note that in the case of quadrupole vibrations the perturbations $\epsilon^{(\pm)}$ of the self-consistent field exhibit the same symmetry and it is natural to assume that $\epsilon^{(\pm)} \sim q_\mu$. On the other hand q_μ and $\epsilon^{(-)}$ have different parity with respect to time-reversal and therefore $\epsilon_{12}^{(-)} = 0$ for vibrations with quadrupole symmetry.* Taking the above statements into account we obtain from (5.26) a system of equations for $\Delta^{(-)}$ and $\epsilon^{(+)}$

$$\Delta_{12}^{(-)} + \sum_{1'2'} \frac{\langle 1\tilde{2}|G|\tilde{2}'1'\rangle}{E_{1'2'}^2 - \omega^2} \{ E\,\xi^{(-)2}\,\Delta^{(-)} - \omega\xi^{(-)}\eta^{(+)}\,\epsilon^{(+)} \}_{1'2'} = 0,$$

$$\epsilon_{12}^{(+)} + \sum_{1'2'} \frac{\overline{\langle 1\tilde{1}'|G|\tilde{2}'2\rangle}}{E_{1'2'}^2 - \omega^2} \{ E\,\eta^{(+)2}\,\epsilon^{(+)} - \omega\,\eta^{(+)}\xi^{(-)}\Delta^{(-)} \}_{1'2'} = 0 . \tag{5.27}$$

---

* This condition is fulfilled automatically for the quadrupole interaction (2.1′). Note that the quadrupole part of the spin-spin interaction $V(1,2) \sim (\vec{\sigma}\,Y^{(2)})_1^{(2)} (\vec{\sigma}\,Y^{(2)})_2^{(2)}$ (see e.g. lectures by Prof. de-Shalit) leads to $\epsilon_{12}^{(+)} = 0$. However, such an interaction is probably of no importance.

Let us now make use of the gauge invariance of the interaction. Assuming $\Delta^{(-)}$ to be a function of the space coordinates and using relation (4.25) we transform the first of equations (5.27) to

$$\sum_{12} \frac{\varphi_2^*(\vec{r})\varphi_1(\vec{r})}{E_{12}^2 - \omega^2} \gamma_2 \left\{ E_{12}\left(\xi_{12}^{(-)2} - \frac{E_{12}^2 - \omega^2}{4 E_1 E_2}\right)\Delta_{12}^{(-)} - \omega \xi_{12}^{(-)} \eta_{12}^{(+)} \epsilon_{12}^{(+)} \right\} = 0 . \qquad (5.28)$$

It is only natural to suppose that $\Delta^{(-)}$ and $\epsilon^{(+)}$ have the same coordinate dependence. Let us put

$$\begin{aligned} \epsilon_{12}^{(+)} &= x q_\mu(12) \\ \Delta_{12}^{(-)} &= y q_\mu(12) \end{aligned} \qquad (5.29)$$

Multiplying (5.28) by $q_\mu(\vec{r})$ and integrating over $\vec{r}$ we obtain after some calculations

$$\sum_{12} |q_\mu(12)|^2 \frac{\Delta^2 E_{12}}{E_1 E_2 (E_{12}^2 - \omega^2)} \{ [((\epsilon_1 - \epsilon_2)/2\Delta)^2 - (\omega/2\Delta)^2] y + (\omega/2\Delta) x \} = 0. \qquad (5.30)$$

Let us now consider the second of equations (5.27). Substituting the quadrupole interaction (2.1') and dropping the small terms (cf. the transformation of (5.5) to (5.6)) we obtain

$$x - \sum_{12} \kappa |q_\mu(12)|^2 \frac{E_{12}}{E_{12}^2 - \omega^2} \{ \eta_{12}^{(+)2} x - (\omega\Delta/2E_1E_2) y \} = 0. \qquad (5.31)$$

Performing identical transformations we can bring (5.30) and (5.31) to the form

$$\begin{aligned} (\omega/2\Delta)\, A(\omega)\, x + C(\omega)\, y &= 0 \\ [(1/\sigma) - 1 - (\omega/2\Delta)^2 B(\omega)] x + (\omega/2\Delta) A(\omega) y &= 0 \end{aligned} \qquad (5.32)$$

where ~~in addition to (5.9)~~

$$A(\omega) = (\kappa/\sigma) \sum_{12} |q_\mu(12)|^2 (\Delta^2 E_{12}/E_1 E_2 (E_{12}^2 - \omega^2)) \qquad (5.33)$$

$$C(\omega) = (\kappa/\sigma) \sum_{12} |q_\mu(12)|^2 (\Delta^2 E_{12}/E_1 E_2 (E_{12}^2 - \omega^2)) \{ ((\epsilon_1 - \epsilon_2)/2\Delta)^2 - (\omega/2\Delta)^2 \}$$

and the quantities σ and $B(\omega)$ have been defined in (5.8) and (5.9). Demanding that the system (5.32) of algebraical equations have a non-trivial solution, we obtain the dispersion equation

$$(\omega/2\Delta)^2 = [(1/\sigma)-1]/[B(\omega) + A^2(\omega)/C(\omega)]. \tag{5.35}$$

In contrast to (5.10) the right-hand side of (5.35) contains an additional term in the denominator. This means that the consideration of the gauge-invariant pairing interaction leads to an additional term in the inertial parameter (in full analogy to the case of rotations).

Expressions similar to (5.17) can easily be obtained for $A(\omega)$ and $C(\omega)$

$$A(\omega) = [g(\ell)]\,; \quad C(\omega) = [\ell^2 g(\ell)]. \tag{5.36}$$

For the first type of vibrations (ω_1 in (5.21)) when $|\ell'| < 1 \ll |\ell''|$ the contribution of distant transitions can be neglected when calculating $A(\omega)$. Contrary to this, the distant transitions give the main contribution to $C(\omega)$ so that in the adiabatic case ($|\ell'| \ll 1$) near transitions can be completely neglected.

There is a marked quantitative difference between the solutions of (5.35) and (5.10) although qualitatively they are the same. Indeed, in the adiabatical approximation the energies of β and γ-vibrations obtained from (5.35) are equal and if non-adiabatic corrections are taken into account $\omega_\beta < \omega_\gamma$, similarly as for equation (5.10). The actual difference between these equations appears when passing over to the hydrodynamical limit. For $\Delta \gg d'$, d'' the inertial parameter $B(\omega)/4\Delta^2$ tends to zero, whereas the limit of the quantity $A^2(\omega)/C(\omega)4\Delta^2$ is finite

$$\frac{A^2(\omega)}{4\Delta^2 C(\omega)} \xrightarrow[\Delta\to\infty]{} \frac{\Sigma_{12}\delta(\epsilon_1)\,|q_\mu(12)|^2}{\sum\limits_{12}\delta(\epsilon_1)\,|q_\mu(12)|^2\,(\epsilon_1-\epsilon_2)^2} \tag{5.37}$$

Thus, by taking into account the gauge-invariant pairing interaction, similarly as in the case of rotations, it is possible to pass over to the hydrodynamical limit.*

5.3. Critical analysis of the results

It can be seen from (5.35) that the vibrational energy depends essentially on the value of the restoring force coefficient ($\sigma^{-1} - 1$). Using the approximation (5.15) for the value σ, the restoring force coefficient is defined by

$$K = (\sigma^{-1} - 1) = (1 - \kappa \sum_{12}\delta(\epsilon_1)\,|q_\mu(12)|^2)/\kappa \sum_{12}\delta(\epsilon_1)\,|q_\mu(12)|^2 \tag{5.38}$$

---

* A similar result has just been obtained by Zaretaky and Urin (private communication).

and is easily seen not to depend on μ. In other word, the restoring force coefficients for β and γ-vibrations are equal to each other. This result is rather surprising from the physical point of view.

Relation (5.38) seems even more doubtful if the restoring forces for spherical and deformed nuclei are compared. The quantity σ'', determined by distant transitions is essentially the same for spherical and deformed nuclei (for β-vibrations cf. (2.19) and (2.30)). The quantity σ', describing the contribution of near transitions is of the same order of magnitude as σ'' for deformed nuclei. For spherical nuclei σ' is sensitive to the filling of the upper shell. In particular, for magic nuclei we have $\sigma' = 0$. Thus we have for the comparison of magic and deformed nuclei

$$\sigma''_{mag} \approx \sigma''_{def} \approx \sigma'_{def} \; ; \qquad \sigma'_{mag} = 0. \tag{5.39}$$

It follows from (5.39) that the restoring force coefficients K (5.38) are of the same order of magnitude for magic and deformed nuclei and differ only by a numerical factor. However this result is known to be wrong. The elasticity of magic nuclei is a volume effect, whereas the elasticity of deformed nuclei is determined by the properties of the surface (we mean the hydrodynamical part defined by (5.38)). Hence the correct ratio is

$$K_{def} / K_{mag} \sim A^{-\frac{1}{3}} . \tag{5.40}$$

Relation (5.40) cannot be fulfilled for any arbitrary value of κ in (5.38) and therefore it is necessary to presume that (5.40) must be considered as some sort of restriction on the choice of the parameter κ. What is the physical meaning of this restriction?

The parameter κ determines the strength of the quadrupole interaction, i.e. the value of the additional self-consistent field with quadrupole anisotropy. The spherically symmetrical part of the self-consistent field has been included directly in the single-particle Hamiltonian $\hat{\epsilon}$. Thus the two parts of the self-consistent field have been taken into account in two different ways. However, the self-consistent field in the nucleus is unique and uniform. In particular, the dividing of the field in a deformed nucleus into the spherical and anisotropic parts is very subjective and it is most certainly not correct to choose the magnitude of the anisotropic part arbitrarily. Thus it is evident from the physical point of view that the value κ must obey a certain condition for the self-consistent field to be uniform. This condition can be very simply formulated in the hydrodynamic limit, where the restoring force coefficient must be determined by surface tension. In other words, the main, volume-dependent, part of the restoring force coefficient K in the hydrodynamic limit must be equal to zero. The hydrodynamic expression for K is given just by (5.38). The condition of the homogeneity of the nuclear self-consistent field can then be formulated as (see 2.32)

$$\kappa \sum_{12} \delta(\epsilon_1) \left| q_\mu(12) \right|^2 \approx \kappa \rho(\mathcal{E}_F) \overline{q^2} = 1 . \tag{5.41}$$

Thus in (5.35) the main term (in the quasi-classical approximation) in the restoring force coefficient is equal to zero and the quantity σ must be calculated up to the next-order corrections to the quasi-classical approximation. These corrections naturally differ for β-and γ-vibrations and the main difference between the energies of β-and γ-vibrations is due to just this factor (since the small difference in the inertial parameters is now of no importance). The corrections to the quasi-classical approximation depend on the single-particle level-scheme and therefore a difference between the β-and γ-vibrational energies can be expected to change non-monotonously as a function of the atomic number A. Existing experimental data seem to confirm this qualitative statement.

6. STRUCTURE OF THE NUCLEON-NUCLEON INTERACTION. VIBRATIONS OF SPHERICAL NUCLEI

6.1. Expansion of the interaction potential in pairing states with a definite angular momentum

Considering various problems connected with the collective properties of nuclei we extracted certain parts of the nucleon-nucleon interaction, those most important in each special case. We divided the interaction matrix elements into two classes, one contributing to the self-consistent field and the other determining the Cooper pairing. Here we shall consider the structure of the nucleon-nucleon interaction in more detail and shall establish the connection between the two types of matrix elements.

We shall first consider the interaction to be described by some potential V. For simplicity we restrict ourselves to central forces. Then we separate the angular dependence of the interaction potential and express it with the help of spherical tensor operators

$$V(\vec{r}_a - \vec{r}_b) = \sum_K v_K(r_a r_b)\,(T^K(a)\,T^K(b))\,. \tag{6.1}$$

The matrix elements of tensor operators T^K between single-particle states $|1\rangle \equiv |n_1 \ell_1 j_1 m_1\rangle$ are equal to

$$\langle 1|T^K_\mu|1'\rangle \equiv [1/(2K+1)^{\frac{1}{2}}]\,(1||T^K||1')\,C^{K\mu}_{1\bar{1}'}, \tag{6.2}$$

where for brevity we have introduced

$$C^{K\mu}_{1\bar{1}'} = (-1)^{j_1' - m_1'}\,(j_1 m_1 j_1' - m_1' | K\mu) \tag{6.3}$$

and $(1||T^K||1')$ is the reduced matrix element, not depending on the magnetic quantum numbers. Taking (6.2) into account the matrix element of the potential (6.1) can be written as

$$\langle 12|V|2'1'\rangle = \sum_k F_K(12;2'1')\,C^{K\mu}_{2\bar{2}'}\,C^{K\mu}_{1'\bar{1}}, \tag{6.4}$$

where

$$F_K(12;2'1') = F^{(K)}(12;2'1')\ (2\|\,T^K\,\|\,2')\ (1'\|\,T^K\,\|\,1) \tag{6.5}$$

and $F^{(K)}$ is a radial Slater integral

$$F^{(K)}(12;2'1') = [1/(2K+1)]\int R_1(a)R_2(b)v_K(r_a r_b)R_{2'}(b)R_{1'}(a)r_a^2 r_b^2 dr_a dr_b. \tag{6.6}$$

Using the expansion (6.4) the interaction Hamiltonian becomes

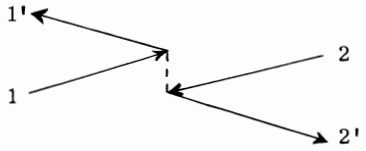
$$H_{int} = \sum_{122'1'} \langle 12|\,V\,|2'1'\rangle\, a_1^+ a_2^+ a_{2'} a_{1'}$$

$$\approx \sum_K \sum_{122'1'} F_K(12;\ 2'1')\ (C^{K\mu}_{2\tilde{2}'}\ a_2^+ a_{2'})\ (C^{K\mu}_{1'\tilde{1}}\ a_1^+ a_{1'}). \tag{6.7}$$

Formula (6.7) represents H_{int} as a product of pair operators

$$\sum_{(m)} C^{K\mu}_{2\tilde{2}'} a_2^+ a_{2'} \tag{6.8}$$

where the pairs consist of a particle and a hole and are characterized by the angular momentum K. The coefficients F_K in this case determine the strength of the interaction between pairs. Putting outgoing or ingoing lines respectively into correspondence with the operators a or a^+ we can represent each term in (6.7) by a Feynman graph



(6.9)

The lines on the same side of the dotted line correspond to operators bound into pairs.

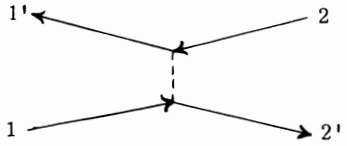
Besides (6.4) three more expansions of the matrix elements are possible, with different grouping into pairs

$$\langle 12|\,V\,|2'1'\rangle = \sum_K \overline{F}_K\ (12;2'1')\ C^{K\mu}_{2\tilde{1}'}\ C^{K\mu}_{2'\tilde{1}} \tag{6.10}$$

$$= \sum_K f_K(12;2'1')\ C^{K\mu}_{12} C^{K\mu}_{1'2'} \tag{6.11}$$

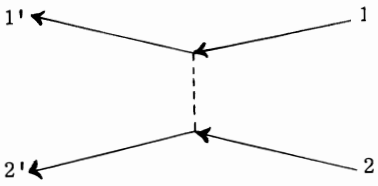
$$= \sum_K \overline{f}_K\ (12;2'1')\ C^{K\mu}_{12} C^{K\mu}_{2'1'} \tag{6.12}$$

(here $C_{12}^{K\mu}$ is the Clebsch-Gordan coefficient $(j_1 m_1 j_2 m_2 \mid K\mu)$. The expansion (6.10), in which the angular momentum of a particle-hole pair is fixed similarly as in (6.4), corresponds to the Feynman graph



(6.13)

In (6.11) and (6.12) two ingoing or two outgoing lines are connected into pairs (particle-particle or hole-hole pair). These expansions correspond to the graph



(6.14)

and can be reduced to each other by a simple permutation of the indices in the Clebsch-Gordan coefficient, so that:

$$\bar{f}_K(12;2'1') = (-1)^{j_1 - j_2 + K - 1} f_K(12;2'1'). \tag{6.15}$$

Using Racah algebra we can connect the coefficients of the various expansions. Thus, in addition to (6.15) we obtain

$$f_K(12;2'1') = \sum_L (-1)^{j_{2'} - j_2 + L} (2L+1) W(Kj_2'j_1 L; j_1'j_2) F_L(12;2'1') \tag{6.16}$$

$$\overline{F_K}(12;2'1') = \sum_L (2L+1)\, W(Kj_1'j_2' L; j_2 j_1) F_L(12;2'1') \tag{6.17}$$

where the W are Racah coefficients. The formulae (6.15 - 6.17) express all the coefficients of the expansions (6.10 - 6.12) with the help of F_K. Let us consider the structure of coefficients F_K (6.5).

6.2. Connection between the expansion coefficients in the asymptotic approximation

The radial part of the single-particle wave-functions depends weakly on the nuclear quantum numbers. The same is true for the radial matrix element $F^{(K)}$ (6.6). In addition, the dependence of $F^{(K)}$ on K is also weak,

because of the short-range character of the forces. (For a δ-function forces $F^{(K)}$ do not depend on K at all). Thus the main dependence in (6.5) on the quantum numbers is due to the reduced matrix elements, the explicit expression for which is

$$(1\|T^K\|2) = (-1)^{j_2-\frac{1}{2}}\left[\frac{(2j_1+1)(2j_2+1)(2\ell_1+1)(2\ell_2+1)}{2K+1}\right]^{\frac{1}{2}} W(\ell_1 j_1 \ell_2 j_2; \tfrac{1}{2} K)(\ell_1 0\, \ell_2 0 | K 0) \tag{6.18}$$

Note that the Clebsch-Gordan coefficient $(\ell_1 0 \ell_2 0|\, K0)$ in (6.18) contains a factor $(-1)^{(\ell_1+k+\ell_2)/2}$ and is thus an alternating quantity for arbitrary values of ℓ_1 and ℓ_2. The collective effects that we are studying are described by coherent quantities, containing sums over wide regions of the levels ℓ_1 and ℓ_2. Hence the main contribution* to collective effects comes from the diagonal quantities $(j\ell\|\, T^K\|\, j\,\ell)$. In addition, the main contributions to the sums over states $|\, j\ell\,\rangle$ come from levels with high angular momenta. Thus, it is possible to use an asymptotic expression, valid for $2j+1 \gg 1$

$$(\ell j\|\, T^K\|\, \ell j) \approx \theta(K)\,(-1)^{\frac{K}{2}} \frac{K!}{2^K\left(\frac{K}{2}!\right)^2}\left(2j+1)^2 - K^2\right)^{\frac{1}{4}} \tag{6.19}$$

where $\theta(K)$ is equal to unity for even K and to zero for odd K. Putting (6.19) into (6.5) we obtain (for $K \ll 2j+1$)

$$F_K(12;21) \approx F^{(K)}\,\theta(K)\sqrt{(2j_1+1)(2j_2+1)}\left[K!/2^K\left(\frac{K}{2}!\right)\right]^2 \tag{6.20}$$

where $F^{(K)} = F^{(K)}(12;21)$. The quantity (6.20) describes coherent particle-hole interaction. From (6.16) and (6.17) we obtain for the coherent parts of the other coefficient

$$f_K(11;22) = \sum_L (-1)^{j_2-j_1+L}(2L+1)W(Kj_2j_1L;j_2j_1)F_L(11;22)$$

$$\tilde{F}_K(12;12) = \sum_L (2L+1)W(Kj_2j_1L;j_2j_1)\,F_L(12;12). \tag{6.21}$$

From the symmetry properties of $(1\|\, T\|\, 2)$ it is easy to obtain the relation

$$F_K(11;22) = (-1)^{j_1-j_2}F_K(12;12) \tag{6.22}$$

making it possible to connect the right-hand sides of (6.21). As a result we obtain

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* A consideration of the general case leads to a complication of the formulas without changing the essence of the matter and so we restrict ourselves to a consideration of diagonal terms only.

$$\tilde{F}_K(12;12) = (-1)^{\ell_1-\ell_2} f_K(11;22) \tag{6.23}$$

and further it is only necessary to consider one of the relations (6.21). The main contribution to the sum (6.21) comes from terms with large L. This enables us to use the asymptotic expression for $F_L(12;12)$. For large values of ℓ, j, L it is easy to obtain

$$(1 \| T \| 2) \approx (-1)^{(\ell_1+L+\ell_2)/2} \theta(\ell_1+L+\ell_2) 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \times \begin{cases} [\tan(\alpha_{12}^L/2)]^{\frac{1}{2}} \\ \pm[\cot(\alpha_{12}^L/2)]^{\frac{1}{2}} \end{cases} \tag{6.24}$$

where the first expression is to be taken for $\Delta j - \Delta\ell \equiv (j_1 - j_2) - (\ell_1 - \ell_2) = 0$ and the second for $\Delta j - \Delta\ell = \pm 1$. The quantity $\alpha_{12}^L (0 \le \alpha_{12}^L \le \pi)$, having the meaning of the "angle" between j_1 and j_2 is defined as

$$\cos \alpha_{12}^L = \frac{L(L+1) - j_1(j_1+1) - j_2(j_2-1)}{2\left(j_1(j_1+1)\, j_2(j_2+1)\right)^{\frac{1}{2}}} . \tag{6.25}$$

Putting (6.24) into (6.5) we find

$$F(12;12) \approx \tilde{F}^{(L)} \theta(\ell_1+\ell_2+L) 2\pi^{-1} \begin{cases} \tan \dfrac{\alpha_{12}^L}{2} \\ \cot \dfrac{\alpha_{12}^L}{2} \end{cases} \tag{6.26}$$

$$\tilde{F}^{(L)} = F^{(L)}(11;22)$$

correspondingly for even and odd values of $\Delta j - \Delta \ell$. To calculate the sum in (6.21) we can also use the asymptotic form for the Racah coefficients

$$W(K j_2 j_1 L; j_2 j_1) \approx \frac{(-1)^K}{\sqrt{(2j_1+1)(2j_2+1)}} P_K(\cos \alpha_{12}^L) \tag{6.27}$$

and then replace the summation over L by an integration according to

$$\Sigma\, \theta(\ell_1+L+\ell_2)(2L+1)\ldots \to \tfrac{1}{4}(2j_1+1)(2j_2+1) \int d \cos \alpha_{12}^L \ldots, . \tag{6.28}$$

Using (6.26), (6.27) and (6.28) we obtain from (6.21)

$$\tilde{F}_K(12;12) = (-1)^K \sqrt{(2j_1+1)(2j_2+1)}\, \frac{1}{2\pi} \int_0^{\pi} \tilde{F}^{(L)} P_K(\cos \alpha^L) \begin{pmatrix} \tan \dfrac{\alpha^L}{2} \\ \cot \dfrac{\alpha^L}{2} \end{pmatrix} d \cos \alpha^L . \tag{6.29}$$

If we neglect the weak dependence of $\widetilde{F}^{(L)}$ on L the integral in (6.29) can be taken in the general form. For even K we have

$$I_K \equiv \frac{1}{\pi}\int_0^{\pi} P_K(\cos\alpha)\cot\frac{\alpha}{2}\,d\cos\alpha = \left[\frac{K!}{2^K\left(\frac{K}{2}!\right)^2}\right]^2. \qquad (6.30)$$

From (6.29) and (6.30) we obtain

$$\widetilde{F}_K(12;12) = \tfrac{1}{2}\sqrt{(2j_1+1)(2j_2+1)}\, I_K \widetilde{F}. \qquad (6.31)$$

Comparing (6.31) and (6.20) and taking into account (6.23) we find for even K the following relations between the expansion coefficients:

$$\widetilde{F}_K(12;12) = (-1)^{\ell_1-\ell_2} f_K(11;22) = \tfrac{1}{2} F_K(12;21). \qquad (6.32)$$

The particle-particle interaction in a state with angular momentum K (taking exchange effects into account) is determined by the quantity (see (6.11), (6.12) and (6.15))

$$\Gamma^{(0)}_{pp}(1;2;K) = \tfrac{1}{2}\left[f_K(11;22) - \overline{f}_K(11;22)\right] = \theta(K) f_K(11;22). \qquad (6.33)$$

For a particle-hole pair the corresponding quantity is defined by a sum of the graphs (6.9) and (6.13)

$$\Gamma^{(0)}_{ph}(1;2;K) = F_K(12;21) - \overline{F}_K(12;12). \qquad (6.34)$$

Combining (6.32), (6.33) and (6.34) we finally obtain

$$(-1)^{\ell_1-\ell_2}\Gamma^{(0)}_{pp}(1;2;K) = \Gamma^{(0)}_{ph}(1;2;K) = \tfrac{1}{2} F_K(12;21). \qquad (6.35)$$

6.3. Short summary and generalization of the results

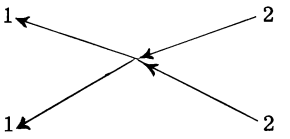
The above calculations were based on two assumptions about the nucleon-nucleon interaction

A. The interaction is described by some potential.

B. The forces are short-range.

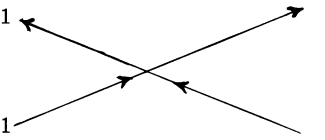
In such a case

(1) The first-order vertex parts for two particles with the angular moment K

$\Gamma_{pp}(11;22;K) =$ 

(6.36)

and those for a particle and a hole

$\Gamma_{ph}(11;22;K) =$ 

(6.37)

are connected by relation (6.35).

(2) The quantities (6.36) and (6.37) for large single-particle angular momenta become separable (see (6.20))

$$\Gamma(11;22;K) = \Gamma(K)\sqrt{2j_1+1}\sqrt{2j_2+1}. \tag{6.38}$$

Note that when more general non-diagonal quantities $\Gamma(11';22';K)$ are considered (see footnote to 6.2), we would obtain analogous results. In particular, instead of (6.38) we can obtain

$$\Gamma(11';22';K) = \Gamma(K)\sqrt[4]{(2j_1+1)(2j_1'+1)}\sqrt[4]{(2j_2+1)(2j_2'+1)}. \tag{6.39}$$

(3) The quantities (6.36) and (6.37) are determined by only one parameter ($F^{(K)} \approx$ const. in (6.20)) in the asymptotic approximation. Note that if we include spin-spin interaction in the potential analogous results can be obtained, but naturally there will be one more parameter.

The following question obviously arises: which of the results obtained are only consequences of the assumptions A and B and which hold in more general cases.

In the generalized Hartree-Fock method which we are considering, the explicit expression for the Hamiltonian $H = H_0 + H_{int}$ is used and the matrix elements of the interaction H_{int} enter directly into the final results. Thus formally this method uses some kind of perturbation theory. However, it can be shown that the final results do not change if we consider the nucleon-nucleon interaction more consistently. In such a case the interaction matrix elements $\langle 12|G|2'1'\rangle$ are replaced by some effective interaction, which is an infinite sum of different Feynman graphs *[18].

In general the quantities (6.36) and (6.37), i.e. the effective interactions of two particles or of a particle and a hole are determined by different sets

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* The most convenient method investigating pairing correlations in a system for such purpose is that of GREEN's functions [18, 19].

of Feynman graphs. Hence relation (6.35) is not satisfied for these quantities. On the other hand the dependence of the effective interaction Γ on the quantum numbers (formula (6.39)) is determined by kinematic factors. Thus (6.39) can be expected to hold also in the general case, but the parameters $\Gamma(K)$ will differ for particle-particle and particle-hole pairs.

The classification of vertex parts as particle-particle and particle-hole ones corresponds to our original division of matrix elements into two classes. It is easy to understand that particle-particle vertex parts cause the Cooper pairing * and that the particle-hole interaction contributes to the self-consistent field. In particular, the quadrupole interaction is determined by the quantity Γ_{ph} $(K = 2)$.

6.4. Vibrations of spherical nuclei

We shall use the results on the structure of the effective interaction in investigating the vibrations of spherical nuclei.

According to the above, we now replace the interaction matrix elements by the effective interaction according to

$$\begin{aligned} \langle 1\tilde{2}|\, G\,|\tilde{2}'1'\rangle &\longrightarrow \sum_K \Gamma_{pp}(12;2'1';K)\, C^{K\mu}_{1\tilde{2}}\, C^{K\mu}_{1'\tilde{2}'} \\ \overline{\langle 1\tilde{1}'|\, G\,|\tilde{2}'2\rangle} &\longrightarrow \sum_K \Gamma_{ph}(12;2'1';K)\, C^{K\mu}_{1\tilde{2}}\, C^{K\mu}_{1'\tilde{2}'}. \end{aligned} \tag{6.40}$$

Putting (6.40) into (1.35) we obtain the following equation for the component $Z^{(\pm)}_{12}(\omega, K)$ with the angular momentum K

$$\omega Z^{(\mp)}_{12}(\omega, K) = E_{12} Z^{(\pm)}_{12}(\omega, K) + \xi^{(\pm)}_{12} \sum_{1'2'} \Gamma_{pp}(12;2'1';K)\xi^{(\pm)}_{1'2'} Z^{(\pm)}_{1'2'}(\omega, K)$$

$$\pm \eta_{12} \sum_{1'2'} \Gamma_{ph}(12;2'1')\, \eta^{(\pm)}_{1'2'} Z^{(\pm)}_{1'2'}. \tag{6.41}$$

We now assume that Γ_{pp} and Γ_{ph} are separable according to (6.39)

$$\begin{aligned} \Gamma_{pp}(12;2'1';K) &= -\Gamma_{pp}(K)\, g_{12}\, g_{1'2'} \\ \Gamma_{ph}(12;2'1';K) &= -\Gamma_{ph}(K)\, g_{12}\, g_{1'2'} \\ g_{12} &= \{(2j_1 + 1)(2j_2 + 1)\}^{\frac{1}{2}}. \end{aligned} \tag{6.42}$$

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* Cooper pairs in spherical nuclei can be shown to be characterized by the total angular momentum $J = 0$. Thus the ground state of a system with Cooper pairing is an analogy (and a generalization) of a state with seniority $v = 0$ in j^n configuration.

Putting (6.42) into (6.41) we obtain, after transformation, the following system of algebraical equations

$$\varphi = \Gamma_{pp} \sum_{12} [\, g^2 \xi^{(+)2} E/(E^2 - \omega^2)]_{12} \varphi + \Gamma_{ph} \sum_{12} [(g^2 \xi^{(+)} \eta^{(+)} E/(E^2 - \omega^2)]_{12} \chi$$

$$+ \omega \Gamma_{pp} \sum_{12} [(g^2 \xi^{(+)} \xi^{(-)} /(E^2 - \omega^2)]_{12} \psi$$

$$\chi = \Gamma_{pp} \sum_{12} [\, g^2 \xi^{(+)} \eta^{(+)} E/(E^2 - \omega^2)]_{12} \varphi + \Gamma_{ph} \sum_{12} [(g^2 \eta^{(+)2} E/(E^2 - \omega^2)]_{12} \chi \tag{6.43}$$

$$+ \omega \Gamma_{pp} \sum_{12} [(g^2 \eta^{(+)} \xi^{(-)} /(E^2 - \omega^2)]_{12} \psi$$

$$\psi = \Gamma_{pp} \sum_{12} [\, g^2 \xi^{(-)2} E/(E^2 - \omega^2)]_{12} \psi + \omega \Gamma_{pp} \sum_{12} [(g^2 \xi^{(-)} \xi^{(+)} /(E^2 - \omega^2)]_{12} \varphi$$

$$+ \omega \Gamma_{ph} \sum_{12} [(g^2 \xi^{(-)} \eta^{(+)} /(E^2 - \omega^2)]_{12} \chi.$$

It is necessary when solving system (6.43) to make certain assumptions about the single-particle level scheme. To make a qualitative analysis of the solutions we shall consider the case of a shell consisting of a system of degenerate states. Then $\xi_{12}^{(+)}$, $\eta_{12}^{(\pm)}$ and E_{12} are constant and the system (6.43) can be greatly simplified. Taking into account that $\xi_{12}^{(-)} = 1$, $\eta_{12}^{(-)} = 0$ 0 and $\xi^{(+)2} + \eta^{(+)2} = 1$ we obtain after simple calculations the following dispersion equation

$$(E^2 - \omega^2)^2 \{ (E^2 - \overline{\Gamma}_{pp}(K))^2 - \omega^2 - (E - \overline{\Gamma}_{pp}(K)) (\overline{\Gamma}_{ph}(K) - \overline{\Gamma}_{pp}(K)) \eta^{(+)2} \} = 0 \tag{6.44}$$

where

$$\overline{\Gamma}(K) = \Gamma(K) \sum_{12} g_{12}^2 . \tag{6.45}$$

From (6.44) we obtain the vibration energies

$$\omega_K = (E - \overline{\Gamma}_{pp}(K)) \sqrt{1 - \frac{\overline{\Gamma}_{ph}(K) - \overline{\Gamma}_{pp}(K)}{E - \overline{\Gamma}_{pp}(K)} \eta^{(+)2}} \tag{6.46}$$

It follows from (6.46) that if the constants $\overline{\Gamma}_{ph}$ and $\overline{\Gamma}_{pp}$ are close to each other, (i.e. if (6.35) is approximately valid), then

$$\omega_K \approx E - \overline{\Gamma}_{pp}(K).$$

Such a spectrum has a simple physical interpretation. The quantity E is the binding energy of a Cooper pair. In an "ideal break-up" of such a pair,

when the particles are infinitely separated after the break-up, the excitation energy of the system is just equal to E. Actually the excitation energy is smaller by the value of the interaction between the particles in the final state (in our case $\overline{\Gamma}_{pp}(K)$)*. Thus for $\overline{\Gamma}_{ph} \approx \overline{\Gamma}_{pp}$ the spectrum is in fact of a single-particle nature.

Collective properties arise if the condition

$$[\overline{\Gamma}_{ph}(K) - \overline{\Gamma}_{pp}(K)] / [E - \overline{\Gamma}_{pp}(K)] \gtrsim 1 \qquad (6.47)$$

is fulfilled. Then the excitation energy can become considerably smaller than the single-particle one. Moreover, in such a case there is a dependence on the shell filling ($\eta^{(+)^2}$)**, and this clearly indicates the collective nature of the excitations.

The quantities $\Gamma(K)$ can be expected to decrease rapidly as K increases. So for a δ-functional interaction (see (6.35) and (6.20)) $\Gamma(2) = \frac{1}{4}\Gamma(0)$) Therefore condition (6.47) can actually be fulfilled for K = 2 provided that

$$\overline{\Gamma}_{ph}(2) \gg \overline{\Gamma}_{pp}(2) . \qquad (6.48)$$

It can now be said with assurance that quadrupole vibrations of the type considered do exist in real nuclei, i.e. that the condition (6.47) is fulfilled for K = 2. If (6.48) is considered to be valid the quantity $\Gamma_{pp}(2)$ can be neglected. Then the equations for quadrupole vibrations will only contain two parameters: $\Gamma_{pp}(0)$ (=E) and $\Gamma_{ph}(2)$, i.e. the interaction of two particles with the angular momentum K = 0 and that of a particle and a hole with K = 2. Note that the model Hamiltonian containing these two types of interaction is just that considered in these lectures. These qualitative arguments give an insight into the reason for the success of quantitative calculations of the spectra of spherical nuclei, based on the simple Hamiltonian with Cooper pairing and quadrupole interaction [8].

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* Note that since $E = \overline{\Gamma}_{pp}(K = 0)$ equation (6.46) for K = 0 gives $\omega = 0$. This is quite natural since Cooper pairs with K = 0 were already taken into account in the ground state.

** The value $\eta^{(+)^2}$ in this case changes from zero for magic nuclei to unity for nuclei with a half-filled shell.

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